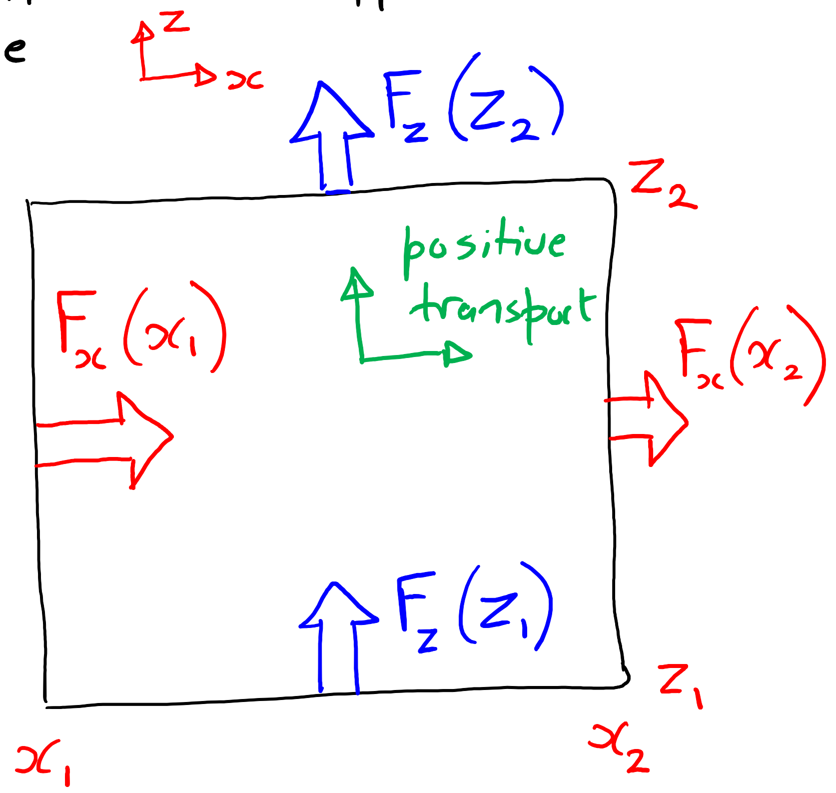
DERIVING A TRANSPORT EQUATION FOR "ϕ"

ϕ is a "volumetric concentration" with units like $kg\,m^{-3}$ or $J\,m^{-3}$ or $\#\,m^{-3}$.
We want to express the idea that ϕ does not just appear or disappear. We introduce an imaginary "control volume" fixed in space



$\Delta x = x_2 - x_1$, $\Delta z = z_2 - z_1$

$\vec{F} = (F_x, F_y, F_z)$ the "flux density" of ϕ
with units like $kg\,m^{-2}\,s^{-1}$ or $J\,m^{-2}\,s^{-1}$

Consider each flux arrow to be an average along its "face".

The total amount of ϕ in the c.v. at time t is $\phi(t)\,\Delta x\,\Delta y\,\Delta z$ (thus ϕ here is implicitly the average conc. within the c.v.)

At a later time $t+\Delta t$ the content of the c.v. has changed by the amount

$$\underbrace{\Delta\phi\ \Delta x\,\Delta y\,\Delta z}_{\text{kg or J or \#}} = \Delta x\,\Delta y\,\Delta z\left[\phi(t+\Delta t) - \phi(t)\right]$$

$$= Q\,\Delta t\,\Delta x\,\Delta y\,\Delta z + \text{net transport}$$

"T"

Q: volumetric production rate, e.g. $kg\,s^{-1}\,m^{-3}$ (source or sink) common to both

$$T = F_x(x_1)\,\Delta t\,\Delta y\,\Delta z - F_x(x_1+\Delta x)\,\Delta t\,\Delta y\,\Delta z$$
$$+\left[F_z(z_1) - F_z(z_1+\Delta z)\right]\Delta t\,\Delta x\,\Delta y$$
$$+\left[F_y(y_1) - F_y(y_1+\Delta y)\right]\Delta t\,\Delta x\,\Delta z$$

Divide through by $\Delta t\,\Delta x\,\Delta y\,\Delta z$

$$\frac{\Delta\phi}{\Delta t} = Q - \frac{F_x(x_1+\Delta x) - F_x(x_1)}{\Delta x} - \frac{\Delta F_y}{\Delta y} - \frac{\Delta F_z}{\Delta z}$$

where $\Delta F_x = F_x(x_1+\Delta x) - F_x(x_1)$

Finally, take limit $\Delta t, \Delta x, \Delta y, \Delta z \longrightarrow 0$

$$\frac{\partial\phi}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} + Q = -\nabla\cdot\vec{F} + Q$$

local tendency

transport

production

(these are all partial derivatives)

Transport mechanisms? Mass Energy

① convection ("bulk flow") $\vec{F}_{conv} = \vec{u}\,\phi$

② diffusion "Fick's Law" $\vec{F}_{diff} = -\mathcal{D}\,\nabla\phi \quad \left[\frac{m^2}{s}\;\frac{1}{m}\;\frac{kg}{m^3}\right]$ ✓

$\mathcal{D}$: molecular diffusivity $m^2 s^{-1}$

① convection $\vec{u}\,\phi$

② conduction $-\mathcal{D}\,\nabla\phi$

③ radiation $\vec{R}$

In the atmos., convection is by far more important than diffusion/conduction

$$* \quad \vec{F}_{diff} = -\mathcal{D}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Recall that "convective transport" is not necessarily "driven" by buoyancy and not necessarily oriented along the vertical

Fourier's Law of Heat Conduction

$$\vec{F}\,[W\,m^{-2}] = -k\,\nabla T$$

$J\,s^{-1}\,m^{-2}$

k: conductivity $W\,m^{-1}\,K^{-1}$