TRANSPORT EQNS FOR WATER VAPOUR, HEAT AND AIR MASS

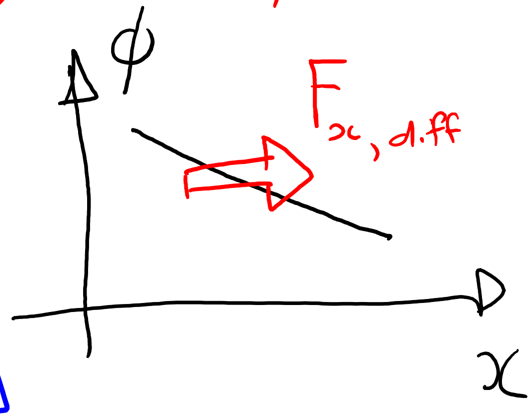
Recall, our flux vector has (potentially) 3 parts, corresponding to the three possible transport mechanisms in the natural world: convection (ie. transport by the bulk flow) + diffusion/conduction (random relative motion at the molecular/atomic level) + (for energy only) radiation.

In the atmosphere, the diffusion/conduction mechanism is of minor effect except very near surfaces (in so-called laminar boundary layers).

For our purposes, Fick's law of diffusion should properly be written

$$\vec{F}_{diff} = -\rho_a \mathcal{D}_{\phi a} \nabla \frac{\phi}{\rho_a}$$

where ρ_a is air density and $\mathcal{D}_{\phi a}$ is the molecular diffusivity of the species "ϕ" in air



A. Conservation of water vapour

$\phi = \rho_v$, absolute humidity

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{F} + Q$$

where Q will allow for phase changes within the air mass

$$\vec{F} = \vec{u}\rho_v - \rho_a \mathcal{D}_{va} \nabla \rho_v/\rho_a \xrightarrow[\text{purposes}]{\text{most}} \vec{u}\rho_v - \mathcal{D}_v \nabla \rho_v$$ ②

$$\frac{\partial \rho_v}{\partial t} = -\nabla.(\vec{u}\rho_v) - \nabla.(-\mathcal{D}_v \nabla \rho_v) = -\nabla.\vec{u}\rho_v + \mathcal{D}_v \nabla.\underline{(\nabla \rho_v)}$$

$$= \overset{Q}{\lambda} - \rho_v \nabla.\vec{u} - \vec{u}.\nabla\rho_v + \mathcal{D}_v \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) \cdot \underline{\left(\frac{\partial \rho_v}{\partial x}, \frac{\partial \rho_v}{\partial y}, \frac{\partial \rho_v}{\partial z}\right)}$$

$$\frac{\partial \rho_v}{\partial t} + \vec{u}.\nabla\rho_v = -\rho_v \nabla.\vec{u} + \mathcal{D}_v \nabla^2 \rho_v + Q$$ advection form

where ∇^2 is the "Laplacian" operator ("diffusion operator"), operates on a scalar, e.g. $\nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$

$$\frac{D\rho_v}{Dt} = -\rho_v \nabla.\vec{u} + \mathcal{D}_v \nabla^2 \rho_v + Q$$

3D velocity divergence

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ in Cartesian coords.

B. Conservation of sensible heat

$\phi = \rho c_p T$, ρ air density, c_p specific heat of air at const. pressure

$\left[\frac{J}{m^3}\right]$ kg m^{-3} , c_p: 1000 J $kg^{-1}K^{-1}$

Fourier law

$$\vec{F} = \rho c_p \vec{u}\, T \boxed{- \rho c_p \kappa \nabla T} + \vec{R}$$

thermal conductivity k, κ thermal diffusivity $[m^2 s^{-1}]$

Treat ρ as const, κ as const, neglect "radiative convergence"

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \vec{F} + Q \qquad \nabla \cdot \vec{R} \approx 0$$

local tendency, flux divergence, source

$$\cancel{\rho}\cancel{c_p}\frac{\partial T}{\partial t} = -\cancel{\rho}\cancel{c_p}\nabla \cdot \vec{u}\, T + \cancel{\rho}\cancel{c_p}\kappa \overbrace{\nabla \cdot \nabla T}^{\nabla^2 T}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = -T\, \nabla \cdot \vec{u} + \kappa \nabla^2 T + Q_T$$

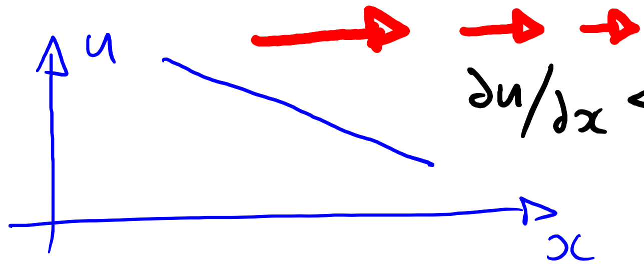
e.g. latent heat of phase change of water

C. Conservation of air mass

$$\phi = \rho \,, \quad \vec{F} = \vec{u}\rho - \rho\, \mathcal{D}_{air/air} \nabla \frac{\rho}{\rho} \,, \quad \nabla 1 = \left(\frac{\partial 1}{\partial x}, \frac{\partial 1}{\partial y}, \frac{\partial 1}{\partial z}\right) = \vec{0}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{u}\rho \qquad \text{flux form}$$

$$\frac{D\rho}{Dt} \equiv \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = -\rho \nabla \cdot \vec{u} \qquad \text{advection form}$$



$\partial u / \partial x < 0$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \vec{u}$$

fractional rate of change of density

Specific volume $\alpha \equiv 1/\rho$

$$\frac{1}{\alpha}\frac{D\alpha}{Dt} = +\nabla \cdot \vec{u}$$

Horizontal divergence

$$\nabla_H \cdot \vec{u}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

if "A" is area on a horiz. plane

$$\frac{1}{A}\frac{DA}{Dt} = \nabla_H \cdot \vec{u}_H$$