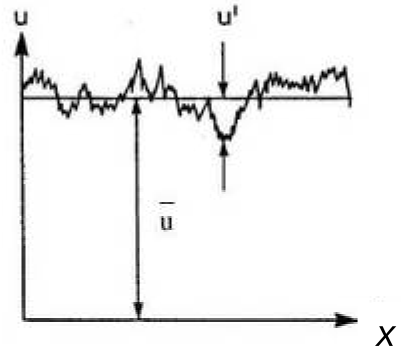
- Horiz. gridpoint spacing for today's NWP is of order 10 km
- Leaving a wide range of scales of motion unresolved
- Implication? – effects of unresolved motion must be parameterized
- Starting point to understand those effects is the "Reynolds decomposition"
- Let $f = f(x, y, z, t)$ and $g = g(x, y, z, t)$ be arbitrary properties
- Divide every dependent variable into the sum of a **resolved** part (which we can regard as a volume average over a grid "cell") and a **residual** (the “fluctuation”):

$$f = \overline{f}(x, y, z, t) + f'(x, y, z, t)$$



- Many notations used for the “mean” (resolved part), e.g. $\overline{f}$, F, $\langle f \rangle$. Here I'll use the overbar
- At best, only *statistics* of the unresolved fields f', g' will be available in NWP – and those are *estimated* by a semi-empirical "*parameterization*"

- Formally, a volume average looks like

$$\bar{f}(x,y,z,t)=\frac{1}{L_x L_y L_z}\int_{x-(L_x)/2}^{x+(L_x)/2}\int_{y-(L_y)/2}^{y+(L_y)/2}\int_{z-(L_z)/2}^{z+(L_z)/2} f(x_*,y_*,z_*,t)\; dx_*\, dy_*\, dz_*$$

- Here x_* (etc.) are dummy variables; L_x would be the gridlength along x. Implicitly, each NWP gridpoint value is an average over a gridcell of volume $L_x L_y L_z$
- Still a function of (x_{IJK}, y_{IJK}, z_{IJK}, t_N), where indices I,J,K label the gridpoints; but presumably a much smoother function than $f=\bar{f}+f'$
- Averaging a product leads to a surprising result: the first step is easy

$g=\bar{g}+g'$ (etc.) so that $f\,g=[\bar{f}+f'][\bar{g}+g']=\bar{f}\,\bar{g}+\bar{f}\,g'+f'\,\bar{g}+f'\,g'$ *

- We simplify this using Reynolds' averaging rules * we need to be able to average quantities like (e.g.) $-\vec{u}\cdot\nabla T$ or $\frac{\partial}{\partial x} uT$

$$\overline{f+g} = \overline{f} + \overline{g}$$

$$\overline{\alpha f} = \alpha \overline{f} \qquad (\alpha \text{ any constant})$$

$$\overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s} \qquad (\text{"}s\text{" being } x \text{ or } y \text{ or } z \text{ or } t\,)$$

$$\overline{\overline{f}\, g} = \overline{f}\,\overline{g}$$

Key STATISTICS of f

Mean square value : $\overline{f^2}$

Variance : $\sigma_f^2 \equiv \overline{(f\,')^2}$

Standard deviation: $\sigma_f \equiv \sqrt{\overline{(f\,')^2}}$

imply

$$\overline{\overline{f}} = \overline{f} \qquad \overline{g\,'} = 0 \qquad \overline{\overline{f}\, g\,'} = \overline{f}\;\overline{g\,'} = 0 \qquad \overline{\overline{f}\;\overline{g}} = \overline{f}\;\overline{g}$$

$$\overline{f\,g} = \overline{[\overline{f} + f\,'\,]\,[\overline{g} + g\,'\,]} = \overline{\overline{f}\;\overline{g} + \overline{f}\,g\,' + f\,'\,\overline{g} + f\,'\,g\,'}$$

$$= \overline{f}\;\overline{g} + \overline{\overline{f}\,g\,'} + \overline{f\,'\,\overline{g}} + \overline{f\,'\,g\,'} = \overline{f}\;\overline{g} + \overline{f\,'\,g\,'}$$

"covariance"

If statistics do not change along x, y axes we say the flow (or system) is "horizontally-homogeneous"

- The instantaneous, local vertical convective flux density of water vapour is

$$E = w\,\rho_v \quad [\mathrm{kg\ m^{-2}\ s^{-1}}]$$

In the ABL $\Big\}$ $\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

- The mean vertical vapour flux density is therefore

$$\overline{E} = \underbrace{\cancel{\overline{w}}}_{0}\,\overline{\rho}_v + \overline{w'\rho_v'}$$

eddy flux of water vapour

$\therefore \underbrace{\cancel{\frac{\partial \overline{u}}{\partial x}}}_{0} + \underbrace{\cancel{\frac{\partial \overline{v}}{\partial y}}}_{0} + \frac{\partial \overline{w}}{\partial z} = 0$ *

* In horiz. homogeneous flow

- First term is vapour transport by the resolved flow; second, by the unresolved flow. Multiplying by the latent heat of vapourization we get the latent heat flux density Q_E, i.e. $Q_E = L_v\,\overline{E}$

$\therefore \frac{d\overline{w}}{dz} = 0, \quad \overline{w}(z) = \overline{w}(0) + C$

$\therefore \overline{w} = 0 \ \forall z$

- Similarly, $\rho\, c_p T \ [\mathrm{J\ m^{-3}}]$ being the volumetric content of sensible heat, the mean convective vertical flux density of sensible heat is

$$Q_H = \rho\, c_p \left[\overline{w}\,\overline{T} + \overline{w'T'}\right]$$

"kinematic" vertical eddy flux of sensible heat

where the $\rho\ c_p$ product has been treated here as (locally) constant

- It remains to show the manner in which the covariances alter evolution of $\overline{\rho}_v, \overline{T}$

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- The zonal momentum per unit volume is: $\rho\, u \quad [\mathrm{kg\, m\, s^{-1}\, m^{-3}}]$

- Multiply by w to get the vertical {zonal} flux density of zonal {vertical} momentum: $\rho\, u\, w$

- Mean (convective) vertical flux density of zonal momentum is therefore

$$\tau_{xz} = \rho \bar{u}\, \bar{w} + \rho \overline{u' w'}$$

eddy mtm flux (or "Reynolds stress")

- Adding a new adjective and dropping others, the "kinematic" momentum flux is

$$\tau_{xz}/\rho = \bar{u}\, \bar{w} + \overline{u' w'}$$

typically negative in ABL (see below)

where the second term is called a "Reynolds stress"

- The "Reynolds stress tensor" R_{ij} is the matrix

$$\begin{bmatrix} \overline{u'^2} & \overline{u' v'} & \overline{u' w'} \\ \overline{v' u'} & \overline{v'^2} & \overline{v' w'} \\ \overline{w' u'} & \overline{w' v'} & \overline{w'^2} \end{bmatrix}$$

Note: $\overline{u'w'} \equiv \overline{w'u'}$ (etc.)

- This can also be called the velocity covariance tensor. Diagonal elements are the velocity variances, $\sigma_u^2 \equiv \overline{u'^2}$ (etc.), where σ_u is the standard deviation of u

- Let's start with an equation for conservation of energy. To keep life simple, let's assume a ~~dry~~ unsaturated system and that the motion is adiabatic and no radiative flux convergence. Then

$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \vec{u}\cdot\nabla\theta = 0$$

- Let's further assume the velocity is non-divergent, $\nabla\cdot\vec{u}=0$. Then since

$\nabla\cdot(\vec{u}\,\theta)=\vec{u}\cdot\nabla\theta+\theta\,\nabla\cdot\vec{u}=\vec{u}\cdot\nabla\theta$ we can put the heat equation in “flux form” as

$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \nabla\cdot(\underline{\underline{\vec{u}\,\theta}}) = \frac{\partial\theta}{\partial t} + \frac{\partial u\theta}{\partial x} + \frac{\partial v\theta}{\partial y} + \frac{\partial w\theta}{\partial z} = 0$$

"kinematic convective flux of potential temp."

- Now apply the Reynolds averaging rules:

$$\frac{\partial\bar{\theta}}{\partial t} + \frac{\partial\overline{u\theta}}{\partial x} + \frac{\partial\overline{v\theta}}{\partial y} + \frac{\partial\overline{w\theta}}{\partial z} = 0$$

where (e.g.) $\overline{w\theta} = \bar{w}\,\bar{\theta} + \overline{w'\theta'}$

- Expand each product,

$$\frac{\partial \overline{\theta}}{\partial t} + \frac{\partial}{\partial x}\left[\overline{u}\,\overline{\theta} + \overline{u'\theta'}\right] + \frac{\partial}{\partial y}\left[\overline{v}\,\overline{\theta} + \overline{v'\theta'}\right] + \frac{\partial}{\partial z}\left[\overline{w}\,\overline{\theta} + \overline{w'\theta'}\right] = 0$$

- To make this look more familiar, convert back to "advection form"

minus the divergence of the eddy (unresolved) heat flux

$$\frac{D\overline{\theta}}{Dt} \equiv \frac{\partial \overline{\theta}}{\partial t} + \overline{u}\,\frac{\partial \overline{\theta}}{\partial x} + \overline{v}\,\frac{\partial \overline{\theta}}{\partial y} + \overline{w}\,\frac{\partial \overline{\theta}}{\partial z} = -\frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}$$

- Finally, we see the influence of the unresolved processes on resolved (mean) θ. Gradients in the unresolved heat fluxes (“eddy heat flux divergence-convergence”) contribute to evolution of the volume-average temperature. The vertical eddy heat flux is particularly important, especially within the turbulent ABL

- Recall the instantaneous *u*-mtm eqn is

$-\nabla\cdot\left(-\nu\nabla u\right)$

Newtonian viscous mtm flux

$$\frac{Du}{Dt} \equiv \frac{\partial u}{\partial t} + \vec{u}\cdot\nabla u = \frac{-1}{\rho}\frac{\partial p}{\partial x} + f\ v + \nu\nabla^2 u$$

"molecular friction"

- Neglecting molecular friction and adopting flux form (under the Boussinesq approximation, density treated as constant and velocity assumed non-divergent),

$$\frac{\partial u}{\partial t} = \frac{-1}{\rho_0}\frac{\partial p}{\partial x} + f\ v - \nabla\cdot\left(\vec{u}\ u\right) = \frac{-1}{\rho_0}\frac{\partial p}{\partial x} + f\ v - \frac{\partial uu}{\partial x} - \frac{\partial vu}{\partial y} - \frac{\partial wu}{\partial z}$$

- After applying the Reynolds average, and bringing advection terms to the lhs,

friction due to unresolved motion

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = \frac{-1}{\rho_0}\frac{\partial \bar{p}}{\partial x} + f\ \bar{v} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{v'u'}}{\partial y} - \frac{\partial \overline{w'u'}}{\partial z}$$

"turbulent friction"

- Gradients in the unresolved (or "eddy") momentum fluxes, i.e. Reynolds stress gradients, result in acceleration (or deceleration) of the resolved velocity

"Rayleigh friction" replaces this with $-\alpha\,\bar{u}$ (a dumb model)

- Will illustrate only the so called "first-order" closure. Taking the vertical eddy flux of heat as an example, by analogy with Fourier's law of conduction we write

"eddy diffusion model" or "K-theory"

$$\overline{w'\theta'} = -K_h \frac{\partial \overline{\theta}}{\partial z}$$

"unresolved convective heat flux density"

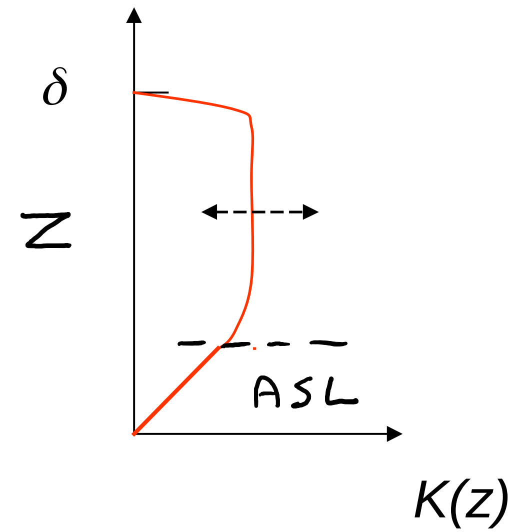
where K_h [m^2 s^{-1}] is the "eddy diffusivity for heat" (very much larger than the molecular thermal diffusivity). The eddy flux is "driven" by the vertical gradient in the resolved (or "mean") potential temperature

- So what is the eddy heat flux in a neutral layer?
- This "parameterization" leaves us with the problem of how to prescribe K_h
- Some empiricism is inevitable
- Analogous closure for Reynolds stresses of form $$\overline{u'w'} = -K_m \left[\frac{\partial \overline{u}}{\partial z} + \frac{\partial \overline{w}}{\partial x} \right]$$

has the needed symmetry

where K_m [m^2 s^{-1}] is the "eddy viscosity" (often set equal to K_h)

- Varies in proportion to *z* near ground
- Increases with increasing mean wind shear
- Increases with increasing thermal instability
- Becomes small at the top (δ) of the ABL



FOR TODAY'S EXERCISE ASSUME

$$\rho \sim 1\ \mathrm{kg\,m^{-3}}, \quad c_p = 10^3\ \mathrm{J\,kg^{-1}\,K^{-1}}$$

Then (e.g.) $\overline{w'\theta'} = 0.3\ \mathrm{K\,m\,s^{-1}} \rightarrow Q_H = 300\ \mathrm{W\,m^{-2}}$

Under what circumstance might we find $\overline{u'w'} < 0$ in the ABL?