Motivation:

- "forecasters do not rely exclusively on computer-generated guidance... they attempt to understand the factors likely to be responsible for the current and the next day's weather." (Durran & Snellman, 1987)

- Let U , L be velocity and length scales characterizing synoptic scale motion. The (dimensionless) Rossby number

$$R_o = \frac{U/L}{f}$$

gives the relative magnitudes of inertial and Coriolis force**

- Wikipedia: "Atmospheric and oceanographic flows take place over horizontal length scales which are very large compared to their vertical length scale, and so they can be described using the shallow water equations... The quasi-geostrophic equations are approximations to the shallow water equations in the limit of small Rossby number, so that inertial forces are an order of magnitude smaller than the Coriolis and pressure forces. If the Rossby number is equal to zero then we recover geostrophic flow."

$$** \quad \underbrace{\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u}}_{U^2/L} = \ldots \underbrace{-\hat{k}\times f\,\vec{u}}_{f\,U} \qquad R_o = \frac{U/L}{f} = \frac{U^2/L}{f\,U}$$

- on the synoptic scale, hydrostatic approx. adequate, $\frac{\partial p}{\partial z} = -\rho\, g$ or $\frac{\partial z}{\partial p} = -\frac{\alpha}{g}$

  and $p_{\mathrm{sfc}} = \int_{z_{\mathrm{sfc}}}^{\infty} \rho\, g\, dz$

- but $\vec{V}_g = \frac{g}{f}\left(-\frac{\partial z}{\partial y},\ \frac{\partial z}{\partial x}\right)$ and $\frac{\partial \omega}{\partial p} = -\nabla_H \cdot \vec{V}_g \equiv -\frac{\partial U_g}{\partial x} - \frac{\partial V_g}{\partial y}$

- so (in effect) density controls the wind: "the mass field controls the wind field"
- thus weather forecasting is largely synonymous with the ability to anticipate how the isobaric height field will *develop*
- gas law, hypsometric eqn & thermal wind eqn also considered part of geostrophic theory
- one might consider this potentially a prognostic theory if one added

$$\frac{D\theta}{Dt} = 0 = \frac{\partial \theta}{\partial t} + \vec{V}_g \cdot \nabla_H \theta + \omega\, \frac{\partial \theta}{\partial p}$$

(unsaturated adiabatic flow of a cloudless atmosphere transparent to radiation)

- updating θ allows to update *T* and then (via gas law) ρ

- the QG eqns predict the rates of change of the height, vertical velocity and vertical vorticity; they are inexact, but provide a $\underbrace{\text{useful interpretive theory}}_{\text{(Assignment 2 is intended to convince you of this)}}$

- the QG eqns are useful because in mid-latitude synoptic scale systems "the fields of vertical motion (ω) and geopotential tendency ($\chi = g\,\partial Z/\partial t$) are primarily determined by the distribution of vorticity advection and thermal advection." **

- QG theory reduces the 5-variable primitive equations to a 1-equation system: all variables (u_g, v_g, ω, p, T) can be obtained from the height field Z. Early NWP models exploited this.

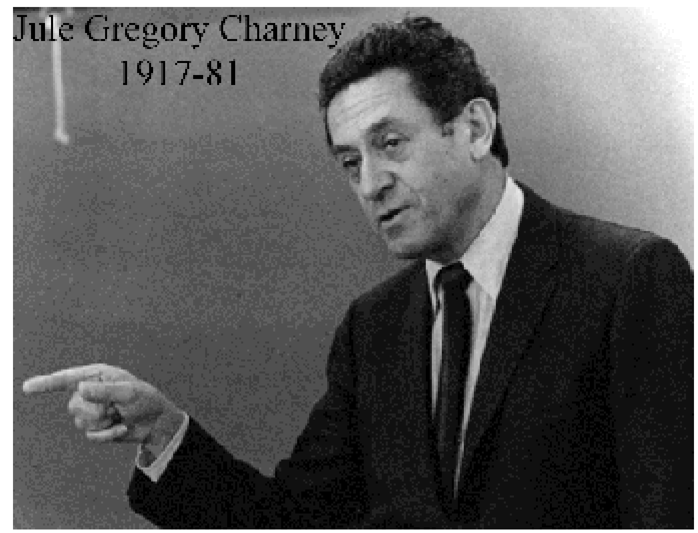
** on the synoptic scale, the atmosphere is considered to be in a state of "delicate imbalance"

- not quite in hydrostatic balance
- not quite in gradient balance

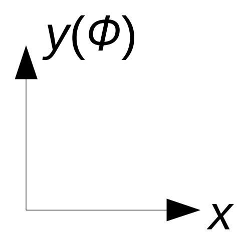
Derivation:

- Rossby, Eady, Charney, Phillips and others: e.g. Charney 1949, On a Physical Basis for Numerical Prediction of Large-Scale Motions in the Atmosphere, J. Meteorol. 6, 371-385.
- emerges from scale analysis; or by an asymptotic expansion** of the governing eqns in the Rossby number $R_o = U/(f\,L)$, *U* and *L* being velocity and length scales characterizing synoptic scale motion

** e.g. $\omega = 0 + R_o\,\omega^{(1)} + R_o^2\,\omega^{(2)} + R_o^3\,\omega^{(3)} + \ldots$



Assumptions/Restrictions/Idealizations:



- frictionless, extra-tropical flow
- hydrostatic approx.
- *β*-plane* approx., $f = f(y) = f_0 + \beta\,y$

**as distinct from the f-plane approximation, wherein f is assumed to be constant* (AMS Glossary)

$$f_0 = 2\,\Omega\,\sin\phi_0$$

$$\beta = (\partial f/\partial y)_{\phi_0} = 2\,\Omega\cos\phi_0/R$$

$f = 2\Omega\sin\phi \Rightarrow \frac{\partial f}{\partial \phi} = 2\Omega\cos\phi$

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \phi} \Big/ \frac{\partial y}{\partial \phi} \sim 1/R$

- decompose horiz. velocity field: $\vec{V} = \vec{V}_g + \vec{V}_{ag}$ where $\vec{V}_g = (u_g, v_g) = \frac{g}{f_0}\,\hat{k}\times\nabla Z$
- neglect vertical advection
- evaluate horizontal advection using the *geostrophic* component (only)

R = earth radius

Δy

$\Delta\phi$

$\Delta y = R\,\Delta\phi$

Some key variables/features/definitions:

"velocity potential" function B

$\vec{u} = \nabla B$

$$\vec{V}_g = \frac{g}{f_0}\hat{k}\times\nabla Z = \frac{g}{f_0}\left(-\frac{\partial Z}{\partial y}, \frac{\partial Z}{\partial x}\right)$$

(*Z* , or gZ/f_0, constitutes a "streamfunction")

(as distinct from a velocity potential function)

$$\nabla_H\cdot\vec{V}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{g}{f_0}\left(-\frac{\partial^2 Z}{\partial x\,\partial y} + \frac{\partial^2 Z}{\partial y\,\partial x}\right) = 0$$

(only the ageostrophic wind component contributes to horiz. divergence)

$$\eta = \zeta_g + \overbrace{f_0 + \beta y}^{f}$$

vertical component of the absolute vorticity

geostrophic relative vorticity

earth vorticity

$$\zeta_g = \hat{k}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ u_g & v_g & 0 \end{vmatrix}$$

$$\zeta_g = \hat{k}\cdot(\nabla_H\times\vec{V}_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{g}{f_0}\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right) = \frac{g}{f_0}\nabla_H^2 Z$$

(Poisson's eqn., easily solved numerically. Vertical vorticity equals Laplacian of the streamfunction. *Z* is "invertible" to give the vorticity and the wind)

$$\frac{D_g}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V}_g\cdot\nabla_H$$

material derivative following the geostrophic wind

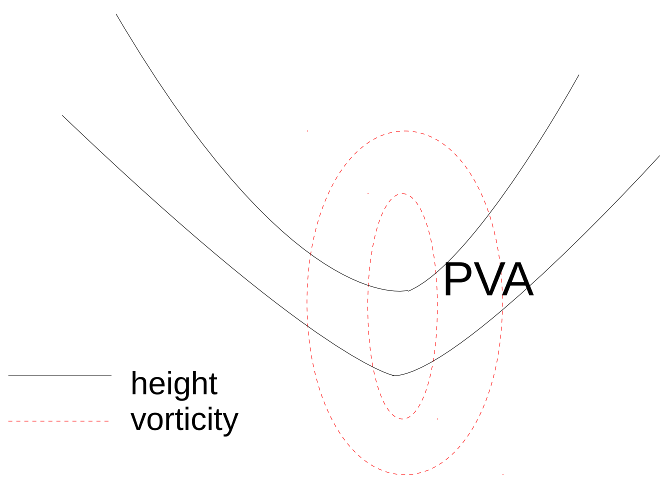
$$\frac{\partial \zeta_g}{\partial t} + \vec{V}_g \cdot \nabla_H (\zeta_g + f) = f_0 \frac{\partial \omega}{\partial p} \qquad (\omega = -\rho\, g\, w)$$

where $\nabla_H \cdot \vec{V}_{ag} + \frac{\partial \omega}{\partial p} = 0$

Alternative statement:

$$\frac{D_g \eta}{Dt} = -f_0 \; \nabla_H \cdot \vec{V}_{ag}$$

absolute vorticity of a parcel varies along its path, as driven by the forcing term on the r.h.s., which represents horizontal convergence/divergence (or equivalently, vertical stretching of the column)



- At the trough axis, local max in cyclonic relative vorticity and thus in abs. vorticity
- As a parcel moves downwind from the trough axis, its vorticity decreases, i.e. $D_g \eta$ /Dt <0, implying $\nabla \cdot \vec{V}_{ag} > 0$
- Using the natural coords, value of $-v \frac{\partial \eta}{\partial s}$ is positive ("PVA")

Vorticity advection signifies horizontal divergence, which is an indicator for $\partial\omega/\partial p \neq 0$, and thus for non-zero ω (vertical motion)

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$$\underbrace{\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]}_{\text{(3D) curvature operator}} \omega = \frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla_H \eta\right] + \frac{R_d}{\sigma\, p}\nabla_H^2\left[\vec{V}_g \cdot \nabla_H T\right]$$

$\nabla_H^2 = \nabla_H \cdot \nabla_H Q$

Scalar

$\nabla_H = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\propto$ height gradient of vorticity advection

$\propto$ Laplacian of temperature advection

where $\sigma = -\frac{R_d T}{p}\frac{\partial \ln\theta}{\partial p}$ $\left[\mathrm{Pa}^{-2}\,\mathrm{m}^2\,\mathrm{s}^{-2}\right]$ is the "static stability" (normally positive and of order 10^{-6} in free atmos.)

strong stable stratification

If $\sigma \to \infty$, $\nabla_H^2 \omega = 0$ (Laplace's eqn) and if ω=0 on boundaries, then ω=0 everywhere

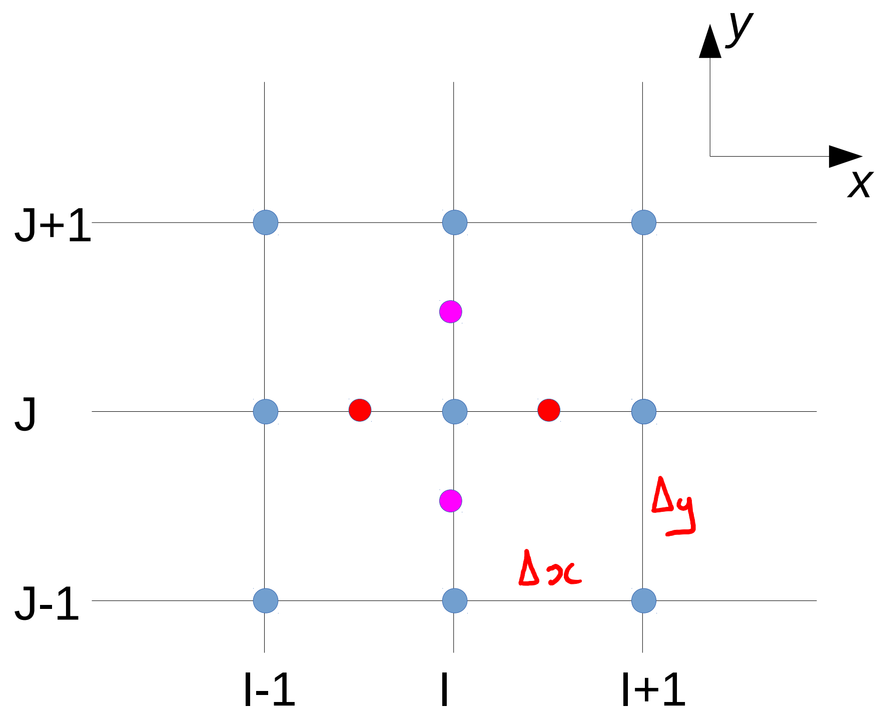
Recall, we define the advective "temperature advection rate" as $A_T = -\vec{V}_g \cdot \nabla_H T$ *

and similarly $A_\eta = -\vec{V}_g \cdot \nabla_H \eta$

$$\frac{dT}{dt} = -\vec{V}_g \cdot \nabla_H T - \ldots$$

Positive temperature advection (PTA) means $A_T > 0$, etc.

In Cartesians, $\nabla_H^2 \omega = \dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}$

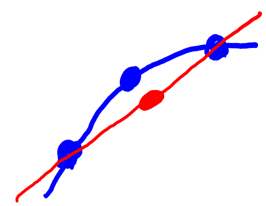


And recall, curvature is slope of slope

$$\frac{\partial^2 \omega}{\partial x^2} \equiv \frac{\partial}{\partial x} \frac{\partial \omega}{\partial x}$$

Evaluate *∂ω/∂x* at red points by finite difference, using known values on the grid:

$$\left(\frac{\partial \omega}{\partial x}\right)_{I-1/2,J} = \frac{\omega_{I,J} - \omega_{I-1,J}}{\Delta x} \qquad \left(\frac{\partial \omega}{\partial x}\right)_{I+1/2,J} = \frac{\omega_{I+1,J} - \omega_{I,J}}{\Delta x}$$



$$\left(\frac{\partial^2 \omega}{\partial x^2}\right)_{I,J} = \frac{\dfrac{\omega_{I+1,J} - \omega_{I,J}}{\Delta x} - \dfrac{\omega_{I,J} - \omega_{I-1,J}}{\Delta x}}{\Delta x} = \frac{\omega_{I+1,J} + \omega_{I-1,J} - 2\omega_{I,J}}{\Delta x^2}$$

Similarly, evaluate *∂ω/∂y* at magenta points by finite difference – and if $\Delta x = \Delta y = \Delta$, then

$$\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)_{I,J} = \frac{\omega_{I+1,J} + \omega_{I-1,J} + \omega_{I,J+1} + \omega_{I,J-1} - 4\omega_{I,J}}{\Delta^2}$$

Laplacian is positive if central value smaller than average of its neighbours

The "envy operator"

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

Laplacian "tries" to produce a linear $T(x)$

$$\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = \frac{-f_0}{\sigma}\frac{\partial A_\eta}{\partial p} - \frac{R_d}{\sigma\, p}\nabla_H^2 A_T$$

height gradient of vorticity advection

Laplacian of temperature advection

- interpret the LHS as (3D) Laplacian of omega, visualized with gridlength Δ

$$\left[\frac{\overline{\omega}_{\mathrm{nbrs}} - \omega}{(\Delta^2)/4}\right] = \frac{-f_0}{\sigma}\frac{\partial A_\eta}{\partial p} + \frac{-R_d}{\sigma\, p}\left[\frac{\overline{A}_{T\,\mathrm{nbrs}} - A_T}{(\Delta^2)/4}\right]$$

A B

units are consistent

- if RHS is positive, central value of omega is smaller than its neighbours – **local updraft**
- to make term A positive, we need A_η to decrease with increasing p, i.e. A_η to increase with increasing height – *strong PVA aloft implies updraft*
- to make term B positive, need A_T exceeding its neighbours – *local "hot spot" for thermal advection implies updraft*

$$\frac{\sigma}{2}\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = -\nabla_H \cdot \vec{Q}$$

If RHS of this eqn is positive (convergent **Q** vectors) expect locally negative ω (ascent)

$$(\omega = -\rho\, g\, w)$$

or

$$\frac{\sigma}{2}\left[\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\omega = -\frac{\partial|\vec{Q}|}{\partial s} - |\vec{Q}|\frac{\partial\beta}{\partial n}$$

divergence of **Q** expressed in the natural coord. system **aligned with isotherms**

or, symbolically

$$\frac{\sigma}{2}\left[\frac{\overline{\omega}_{\mathrm{nbrs}} - \omega_{\mathrm{P}}}{(\Delta^2)/4}\right] \approx -\nabla_H \cdot \vec{Q}$$

Note: here giving Martin's Eqs (6.40, 6.44). Holton's Eq 6.53 includes an extra term that vanishes in the f = const. approx.

where

$$\vec{Q} = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[\hat{k}\times\frac{\partial\vec{V}_g}{\partial s}\right] = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[-\hat{s}\frac{\partial v_g}{\partial s} + \hat{n}\frac{\partial u_g}{\partial s}\right]$$

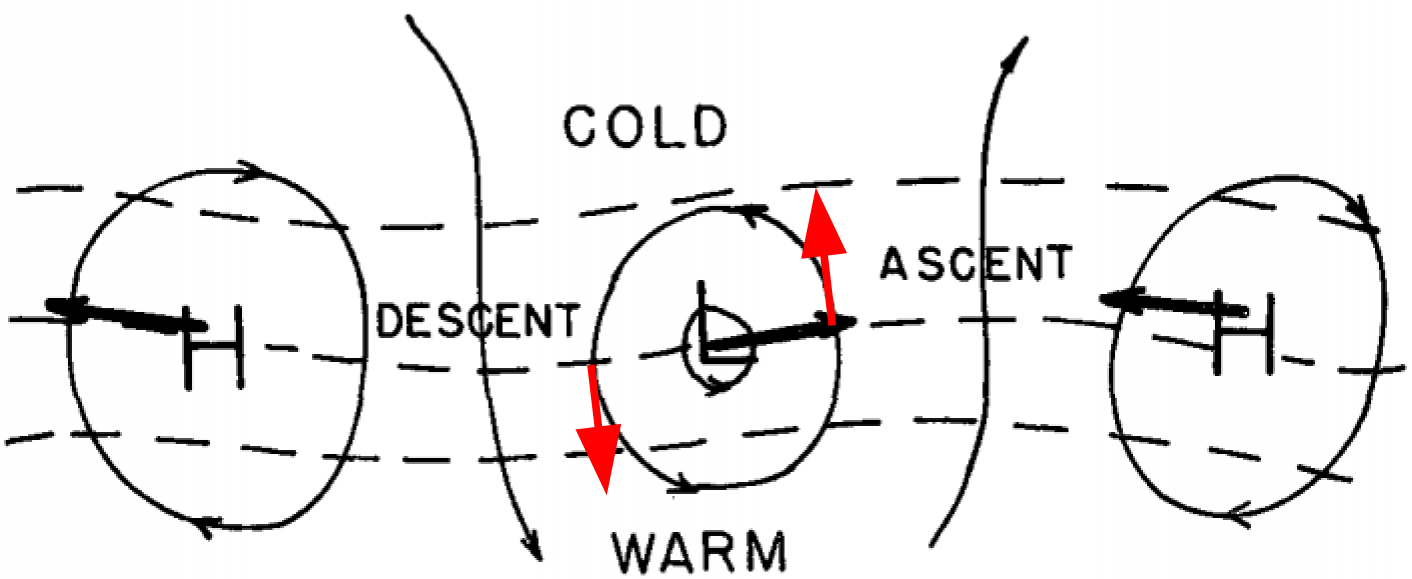
"The **Q** vector can be obtained by evaluating the vectorial change of $\mathbf{V}_g$ along the isotherm (with cold air on the left), rotating this change vector by 90° clockwise, and multiplying the resulting vector by $|\partial T/\partial n|$." (Holton)

Wherever the pattern of the **Q** vector is convergent $(\nabla_H \cdot \vec{Q} < 0)$,

omega is less positive (more negative) at P than in the neighbourhood... **local ascent**

$$\vec{Q} = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[\hat{k}\times\frac{\partial \vec{V}_g}{\partial s}\right] = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[-\hat{s}\,\frac{\partial v_g}{\partial s} + \hat{n}\,\frac{\partial u_g}{\partial s}\right]$$

Here (u_g,v_g) are the components of $\mathbf{V}_g$ in a coordinate system aligned with *isotherms*



- Closed isobars of MSLP (solid)
- isotherms (dashed)
- geostrophic wind vectors
- **Q vectors**

Modified from Sanders & Hoskins (1990; Weather & Forecasting Vol. 5, Fig. 3)

West of the low, v_g is northerly (negative), east of the low it is southerly (positive) so that $\partial v_g/\partial s$ >0; but u_g is zero on west and east sides). Thus only the term along **s** contributes, and the two negatives cancel. At the low, **Q** points eastward. Applying a parallel reasoning to the highs, we conclude **Q** points *westward* , resulting in convergent **Q** vectors on the east side of the low.

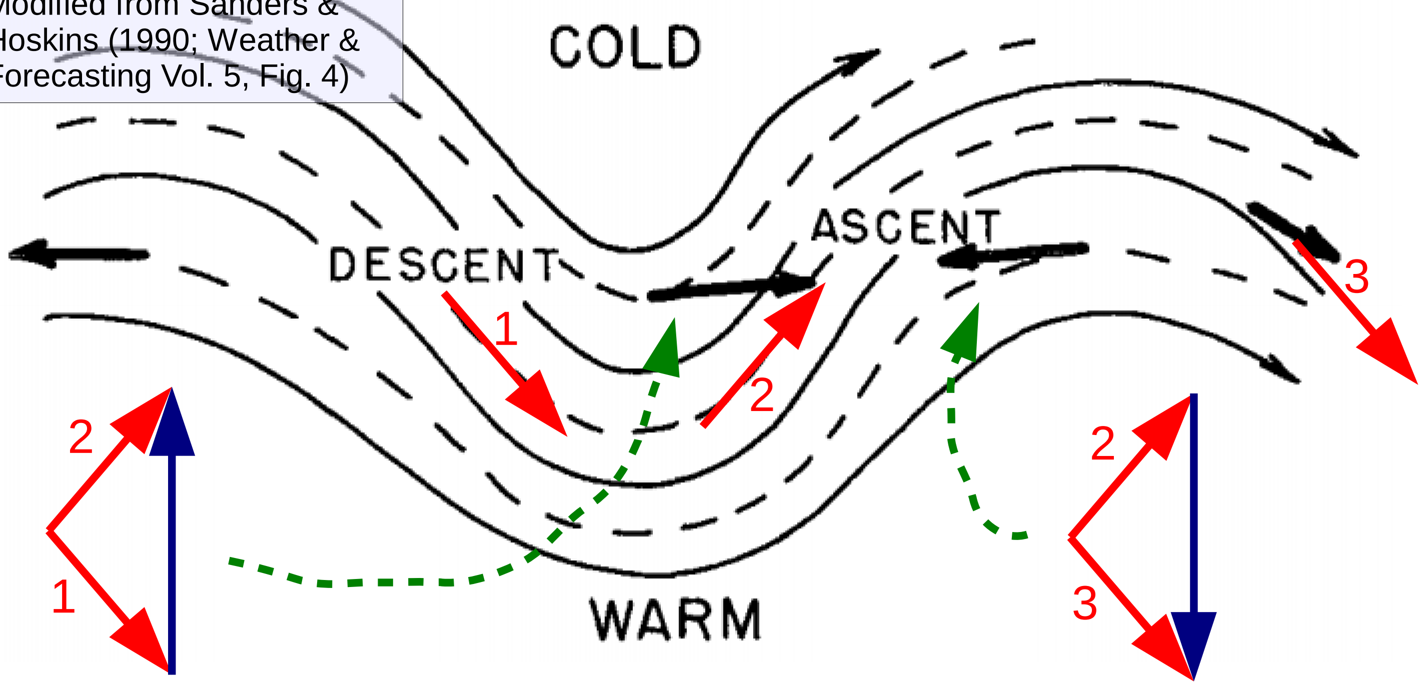
"The **Q** vector can be obtained by evaluating the vectorial change of $\mathbf{V}_g$ along the isotherm (with cold air on the left), rotating this change vector by 90° clockwise, and multiplying the resulting vector by $|\partial T/\partial n|$." (Holton)

$$\frac{\sigma}{2}\left[\frac{\overline{\omega}_{\mathrm{nbrs}}-\omega_{\mathrm{P}}}{(\Delta^2)/4}\right] \approx -\nabla_H\cdot\vec{Q}$$

$$\vec{Q} = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[\hat{k}\times\frac{\partial \vec{V}_g}{\partial s}\right] = \frac{-R}{p}\left|\frac{\partial T}{\partial n}\right|\left[-\hat{s}\,\frac{\partial v_g}{\partial s} + \hat{n}\,\frac{\partial u_g}{\partial s}\right]$$

Here (u_g,v_g) are the components of $\mathbf{V}_g$ in a coordinate system aligned with *isotherms*

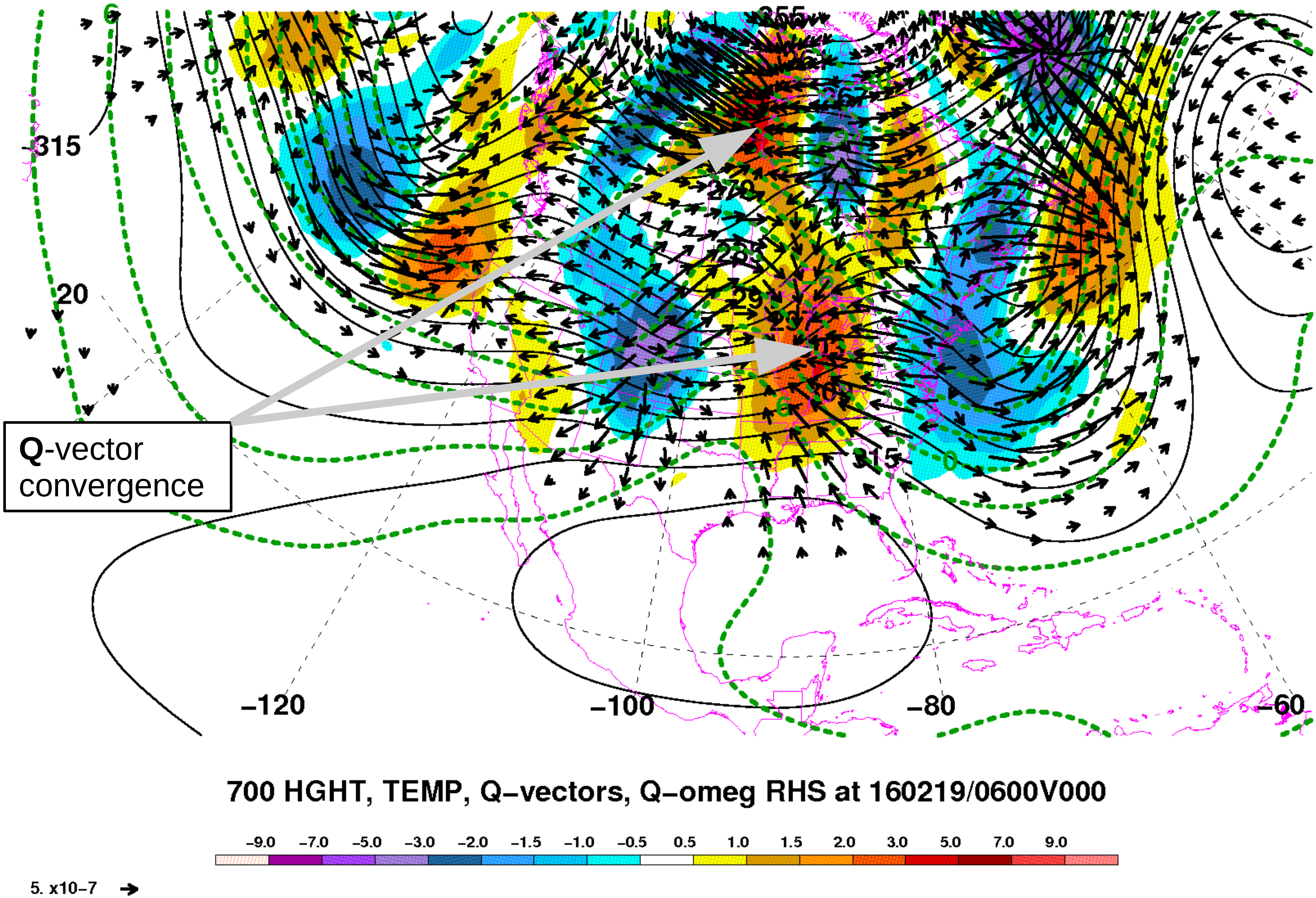
Modified from Sanders & Hoskins (1990; Weather & Forecasting Vol. 5, Fig. 4)



- Height contours (solid)
- isotherms (dashed)
- geostrophic wind vectors
- **Q vectors**

Another idealized example (upper trough & ridge). Note the absence of thermal advection (isotherms parallel to height contours). The v_g component of the geostrophic wind, i.e. the component normal to the isotherms, is zero everywhere. The u_g component, i.e. the component projected onto the (curvy) **s** axis is constant in magnitude, but its orientation changes; that change (across the trough) is given by the **blue vector**, which we rotate 90^o clockwise to get **Q** (a westerly) in the trough. Parallel reasoning implies an easterly in the ridge

"The **Q** vector can be obtained by evaluating the vectorial change of $\mathbf{V}_g$ along the isotherm (with cold air on the left), rotating this change vector by 90^o clockwise, and multiplying the resulting vector by $|\partial T/\partial n|$." (Holton)


Q-vector
convergence
315
20
–120
–100
–80
–60
700 HGHT, TEMP, Q–vectors, Q–omeg RHS at 160219/0600V000
–9.0 –7.0 –5.0 –3.0 –2.0 –1.5 –1.0 –0.5 0.5 1.0 1.5 2.0 3.0 5.0 7.0 9.0
5. x10–7

co-located with **Q-vector convergence** are **high cloud** and **ascent**