EAS 471 Second Lab (unscored) 2014

Numerical solution of 1D heat equation

Consider a copper rod of length $L = 1$ m lying along the x-axis ($-L/2 \leq x \leq L$). The density, heat capacity and thermal diffusivity of copper are approximately

$$\begin{aligned}\rho &= 8900\,[\mathrm{kg\,m^{-3}}]\,,\\ c &= 385\,[\mathrm{J\,kg^{-1}\,K^{-1}}]\,,\\ \kappa &= 2.52\times 10^{-4}\,[\mathrm{m^2\,s^{-1}}].\end{aligned}$$

We suppose that the rod is enclosed in a perfect insulator such that its temperature $T = T(x,t)$, i.e. T may vary along x and in time t but there can be are no radial gradient. It follows that temperature in the rod satisfies the 1D heat equation

$$\frac{\partial T}{\partial t} = \kappa\,\frac{\partial^2 T}{\partial x^2} \tag{1}$$

(this can be obtained from the generalized conservation equation by the substitutions $\theta = \rho\,c\,T$ and $F = -\,\rho\,c\,\kappa\,\partial T/\partial x$, where F is the conductive heat flux).

The temperature of each end of the rod is constrained such that

$$T(L/2,t) = T(-L/2,t) = 0 \tag{2}$$

and initially the temperature satisfies

$$T(x,0) = T_{00}\,\cos\left((m+1)\,\pi\,\frac{x}{L}\right) \tag{3}$$

where $T_{00} = 1$ and m is an even integer.

We notice that the physical variables of this problem imply a time constant

$$\tau = L^2/\kappa = 3976\,[\mathrm{s}] \ , \tag{4}$$

or a little over an hour.

Task

Solve this equation numerically with time steps $\Delta t \ll \tau$ out to an end time $t = 1000$ s, for the cases $m = 0, 2, 4$. Plot your solutions at time intervals of $\tau/3$.

Method

Implement the explicit algorithm that results when Eq. (1) is discretized using a forward time difference and a central space difference,

$$\frac{T_I^{n+1} - T_I^n}{\Delta t} = \kappa \, \frac{T_{I+1}^n + T_{I-1}^n - 2T_I^n}{\Delta x^2} \tag{5}$$

where n indexes the time axis such that $t(n) = n\Delta t$, and I indexes the space axis

$$x(I) = \frac{-L}{2} + (I-1)\,\frac{L}{I_{mx} - 1} \tag{6}$$

such that $\Delta x \equiv L/(I_{mx} - 1)$. Eq. (6) is contrived to give a node (i.e. grid-point) at each end of the rod; you can juggle the value of I_{mx} to control the size of Δx, and thus the total number of nodes along your rod. We can rearrange Eq. (5) to isolate temperatures at the "new" time $t(n+1)$ as

$$T_I^{n+1} = T_I^n + \frac{\kappa\,\Delta t}{\Delta x^2}\,\left(T_{I+1}^n + T_{I-1}^n - 2T_I^n\right) \ . \tag{7}$$

Note that you can expect your algorithm to be numerically unstable if your "diffusion number" $\kappa\Delta t/\Delta x^2$ (the coefficient on the r.h.s. of Eq. 7) exceeds 1/2. Search for, and document, this behaviour.

Briefly document your lab exercise and submit to Ginny for feedback.