EAS 471 2^{nd} Scored Assignment Due: 14 Mar. 2014

Conduction in the plane

The heat equation for a steady state problem in two dimensions may be written

$$\frac{\partial T}{\partial t} = 0 = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T + Q \,, \tag{1}$$

where Q is the volumetric source distribution. We shall suppose there is a steady heat source localized at the origin[1], such that Eq. (1), which incidentally could be named a "Poisson's equation," becomes

$$0 = \kappa \, \nabla^2 T + \alpha \, \delta(x-0) \, \delta(y-0) \, . \tag{2}$$

This gives the steady-state temperature field $T(x, y)$ resulting from the source (whose strength is α). Let the domain be the unit square $-1/2 \le x, y \le 1/2$ and let the boundary temperature be $T = 0^{\circ}$ C. For simplicity let the thermal diffusivity $\kappa = 1$ and let the heat source strength $\alpha = 1$.

Solve this Poisson's equation using an iterative "relaxation method" and the standard computational molecule for the 2-d Laplacian. Set up your grid with uniform mesh $\Delta x = \Delta y = \Delta$ such that gridpoint $i = j = 0$ falls at the origin: only at this gridpoint is the source term (numerical delta function) non-zero, indeed its value at the origin is $1/\Delta^2$. Study the impact of two choices for the resolution, viz. $\Delta = (0.05, 0.25)$. Compare your numerical

[1] Note that the δ-function has units: for present purposes we may define the delta-function according to: $\int_{-\infty}^{\infty} f(x) \, \delta(x - x_1) \, dx = f(x_1)$

solutions with the analytical solution[2],

$$T(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{4\,\pi^{-2}}{(2m+1)^2 + \ (2n+1)^2}\ \cos\left[(2m+1)\,\pi x\right]\ \cos\left[(2n+1)\,\pi y\right] \tag{3}$$

(carrying the summation to $m, n = 100$ will more than suffice, since these limits evidently correspond to very high wave number contributions to the solution).

Iterate your solution until the largest residual $\max(|R_{i,j}|) \leq 10^{-7}$. Remember you may plot your analytical solution with arbitrarily high spatial resolution, ie. as a line, for it is a spatially-continuous solution. However the numerical solution is known (and should therefore be plotted) *only at the gridpoints*.

[2]The solution has here been obtained using a boundary-weighted Galerkin method. Cosine functions have been chosen as basis functions, for they satisfy the required symmetry about the origin, and vanish at the boundary of the unit square; and in view of the use of cosine functions, one can equivalently view the solution as stemming from the Fourier series approach. Note also that the analytical solution can be said to be the Green's function for the Laplacian in the unit square and for the particular boundary condition assumed.