EAS471, "Atmospheric Modelling" Exam 28 April, 2005

Professor: J.D. Wilson Time available: 120 mins Value: 35%

Please answer in the exam booklet. Some pertinent equations/data are given at the back.

Multi-choice (15 x 1% = 15%)

1. If vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ have representation $\overrightarrow{u} = (\alpha, 0, 1)$ and $\overrightarrow{v} = (0, 1, \beta)$ when projected onto orthogonal unit vectors $(\hat{i}, \hat{j}, \hat{k})$, then their inner product $\overrightarrow{u} \bullet \overrightarrow{v} =$ ____

   (a) 0
   (b) 1
   (c) α
   (d) β ✓✓
   (e) $\alpha\beta$

2. α_i and τ_{ij} are respectively rank 1 and rank 2 (Cartesian) tensors in a 2-D space. Under the summation convention the quantity $\alpha_i\, \partial\tau_{ij}/\partial x_j$ expands to

   (a) $\alpha_1\, \partial\tau_{12}/\partial x_1$
   (b) $\alpha_2\, \partial\tau_{21}/\partial x_2$
   (c) $\alpha_1\, \partial\tau_{11}/\partial x_1 + \alpha_1\, \partial\tau_{12}/\partial x_2 + \alpha_2\, \partial\tau_{21}/\partial x_1 + \alpha_2\, \partial\tau_{22}/\partial x_2$ ✓✓
   (d) $(\alpha_1 - \alpha_2)\ (\partial\tau_{11}/\partial x_1 + \partial\tau_{22}/\partial x_2\ )$
   (e) $(\alpha_2 - \alpha_1)\ (\partial\tau_{11}/\partial x_1 + \partial\tau_{22}/\partial x_2\ )$

3. The Von Neumann stability analysis of a finite-difference discretization scheme covers ____ but does not account for ____

   (a) any d.e. (differential eq'n) expressed in Cartesian coordinates; time-dependence
   (b) any linear d.e.; boundary conditions ✓✓
   (c) first- or second-order d.e.'s only; non-linearity
   (d) any linear d.e.; influence of unresolved scale of motion
   (e) heat/diffusion type equations; inhomogeneities (Q- type terms)

4. The von Neumann analysis of $\frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x}$ results in a criterion on the ____ number

(a) Diffusion

(b) Reynolds

(c) Richardson

(d) Courant ✓✓

(e) Euler

5. Which name(s) is/are associated with numerical schemes for solving the heat equation

(a) Crank and Nicolson

(b) Richardson

(c) Dufort

(d) Frankel

(e) all of the above ✓✓

6. The shortest wave that can be represented on a grid with spacing Δx has wavelength

(a) $\pi/\Delta x$

(b) $2\pi/\Delta x$

(c) $\Delta x/2$

(d) Δx

(e) $2\Delta x$ ✓✓

7. If $S_x(f)$ is the power spectrum of a random variable $x(t)$, then $\int_0^\infty S_x(f)\, df =$ ____

(a) 0

(b) ∞

(c) Δx, the grid interval

(d) Δt, the time step

(e) σ_x^2, the variance of $x(t)$ ✓✓

8. If the Fourier transform of $T(x,t)$ satisfies $\partial/\partial t\ \widetilde{T}(k,t) = -K\ k^2\ \widetilde{T}(k,t)$ where k is wavenumber and K is a constant, then T must be governed by a/n ____

(a) advection equation

(b) diffusion equation ✓✓(1)

(c) advection-diffusion equation

(d) time-independent equation ✓✓(1/2)

(e) inhomogeneity

9. In relation to the above question, the Fourier mode $\widetilde{T}(k,t)$ must ____

   (a) grow logarithmically in proportion to (Kk^2t)
   (b) grow exponentially in proportion to Kk^2t
   (c) decay in proportion to $\sqrt{Kk^2t}$
   (d) decay exponentially in proportion to Kk^2t ✓✓
   (e) vanish

10. Suppose $L[\phi(x)] = 0$ (where $L[\ ]$ is a differential operator) is to be solved on $a \leq x \leq b$ using the Bubnov-Galerkin method with basis functions $\theta_j(x)$. The expansion coefficients a_j are optimized by requiring that ____

    (a) $a_i a_j = \delta_{ij}$
    (b) the residual $e(x) = L\left[\sum_{j=1}^{N} a_j\, \theta_j(x)\right]$ must vanish
    (c) the residual $e(x)$ must be minimal
    (d) the residual $e(x)$ must be uncorrelated with each $\theta_j(x)$ over $a \leq x \leq b$ ✓✓
    (e) all of the above

11. Q_c is the mean vertical flux density of an admixture 'c' across the atmospheric surface layer (friction velocity u_*, Obukhov length L). According to the Monin-Obukhov similarity theory, statistics of c (like its standard deviation, σ_c) scale with ____

    (a) u_*
    (b) L
    (c) $u_* L$
    (d) Q_c/L
    (e) Q_c/u_* ✓✓

12. Consider the kinetic energy k of the unresolved motion in a horizontally-uniform layer of the atmosphere that is in local equilibrium (conservation equation given at back): if the layer is stably stratified then buoyant production ____

    (a) tends to increase k
    (b) has the same sign as shear production
    (c) vanishes
    (d) balances shear production
    (e) when summed with shear production, balances viscous dissipation ϵ ✓✓

13. The hemispheric irradiance across a unit of area whose normal is oriented vertically (ie. parallel to unit vector $\hat{k}$) is given by $F_z = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} I(\hat{s})\ (\hat{k} \bullet \hat{s})\ \sin\phi\ d\theta\ d\phi$ where $\hat{k} \bullet \hat{s} = \cos\phi$. If the radiation intensity I is isotropic then $F_z =$ ____

   (a) 0
   (b) $1/2\ \sin(2\phi)$
   (c) $\pi\ I$ ✓✓
   (d) $2\pi\ I$
   (e) $4\pi\ I$

14. Let $X(t)$ denote the displacement (along direction x) of a particle from its point of release at time $t = 0$. Assuming the motion is "diffusive" (with diffusivity K) then (over an ensemble of trials) the statistic $\sigma_x^2(t) \equiv \overline{\left(X(t) - \overline{X}(t)\right)^2} =$ ____

   (a) 0
   (b) $\sqrt{2K\ t}$
   (c) $2K\ t$ ✓✓
   (d) $(2K\ t)^2$
   (e) ∞

15. The colour of the (unpolluted, cloudless) sky is explained by the ____

   (a) wavelength (λ) selectivity of the Rayleigh scattering function
   (b) anisotropy of the Rayleigh scattering function
   (c) wavelength selectivity of the Mie scattering function
   (d) anisotropy of the Mie scattering function
   (e) wavelength-dependence of the (Rayleigh) single scattering albedo ($\omega_o \propto \lambda^{-4}$) ✓✓

Short answer: 4 x 5% = 20%

Answer any **four** questions from this section, writing a (**maximum**) of one page per question. Additional pages used for calculations/working/preparation will not be marked. (**Turn to the back for answers**)

1. Explain the meaning and implications of the term "neutrally-stratified" as applied to the atmospheric boundary-layer, making reference both to the mean vertical flux density of sensible heat (Q_H) and the vertical temperature (or potential temperature) gradient. You may assume a dry atmosphere. Explore the implications of neutrality

with regard to the kinetic energy budget (to this end you may assume steady state and local equilibrium apply; the equations given as data should help).

2. In the symbolic computing language of your choice, write down a code to evaluate the formula

$$T(y) = \sum_{i=1}^{20} \sum_{j=i}^{20} (-1)^{i+j} \sin(\pi y)$$

for a set of real values of y stored in the array $x(k)$, viz. $y = x(k),\ k = (1, 2, ...10)$. Your code need not be executable and need not be in perfect syntax, but all major tasks except I/O must be addressed, including declaration (typing) of the variables.

3. Suppose $\phi = \phi(x, y)$ is defined in the quadrant $x, z \geq 0$ where it is governed by the differential equation (d.e.)

$$U \frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial z^2}$$

($U > 0$ is a real constant) and that this equation is to be solved on a uniform grid whose nodes (i, j) lie at $x(i) = (i-1)\Delta$, $z(j) = (j-1)\Delta$. We may let ϕ_{ij}^n denote the n^{th} iteration towards the solution away from a first guess field ϕ_{ij}^0.

   Discuss the need for (as yet unspecified) boundary conditions in order to fully pose (specify) this problem. Explain whether, if we wished to compute the solution for all $0 \leq x \leq L$, it would be adequate to invoke a single boundary condition on the x-axis at $x = L$, ie. specify $\phi(L, z)$? Classify the terms in the d.e., and address whether (and why) each of the spatial axes is 'one-way' or 'two-way'.

   If we impose $(\partial \phi / \partial z) = 1$ along $z = 0$ then after the 4^{th} iteration the influence of this boundary condition will have propogated along the $z(j)$ axis as far as which gridpoint (j), and why?

4. The lowest order finite difference approximation for $\partial^4 \phi / \partial x^4$ is

$$\frac{\partial^4 \phi}{\partial x^4} = \frac{\phi(x+2\Delta) - 8\phi(x+\Delta) + 20\phi(x) - 8\phi(x-\Delta) + \phi(x-2\Delta)}{\Delta^4} + \mathrm{O}(\Delta^2)$$

   Without worrying about whether it will be numerically stable, formulate an explicit algorithm to solve the differential equation

$$\frac{\partial \phi}{\partial t} = a \, \frac{\partial^4 \phi}{\partial x^4}$$

(where t is time and x is a spatial coordinate). Your algorithm need only be valid at 'interior' gridpoints, ie. those distant no less than 2Δ from the edges of the domain.

Suppose instead you formulated an *implicit* discretization such that (in matrix representation) your scheme for advancing the solution over one timestep was $\phi^{\mathbf{n+1}} \;=\; \mathbf{A}\,\phi^{\mathbf{n}}$. Explain how many non-zero diagonals would appear in the coefficient matrix $\mathbf{A}$.

5. In a typical weather model whose lowest gridpoint lies in the atmospheric surface layer at $z = z_a$, the surface-atmosphere flux of heat (Q_{H0}) is computed as

$$Q_{H0} = \; \rho c_p \, C_M \, C_H \, u_a \; (T_0 - \; T_a)$$

Explain the logic of this parameterization, defining all the terms in the above equation. Explain, too, how coefficients C_M, C_H are "calibrated."

6. Using the control volume method, discretize the differential equation

$$0 = \; \frac{d}{dz} \, \left(K(z) \, \frac{dC}{dz} \right) + \; q_1 \, \delta(z-h)$$

on the domain $0 \leq z \leq L$ (where $L >> h$), assuming boundary conditions

$$\begin{aligned} C(L) \; &= \; 0 \\ \left(- \, K \, \frac{dC}{dz} \right)_{(z=0)} \; &= \; q_2 \end{aligned}$$

(q_1, q_2 are constants that are not necessarily dimensionless). State the neighbour coefficients for an arbitrary internal control volume (ie. not at or adjacent to a boundary), and give a diagram of your grid (including uppermost and lowermost control volumes) that clarifies your procedure. Give also a physical interpretation of the governing equation, ie. explain speculatively what physical problem it models.

Data:

- $$\frac{k_v z}{u_*}\,\frac{\partial \overline{u}}{\partial z} = \phi_m\left(\frac{z}{L}\right) \tag{7}$$

  Dimensionless mean wind shear in the horizontally-uniform atmospheric surface layer, given by the universal Monin-Obukhov function on the r.h.s. (which in neutral stratification, ie. at $|z/L| = 0$, has the limit $\phi_m(0) = 1$). Similar expressions apply for gradients in mean temperature, humidity, etc.

- $$L = \frac{-\,u_*^3}{k_v\,\frac{g}{T_0}\,\overline{w'T'}}$$

  Definition of the Obukhov length, where: u_* is the friction velocity, k_v is the von Karman constant, T_0 [K] is the mean temperature of the atmospheric surface layer, and $\overline{w'T'}$ [$\equiv Q_H/(\rho c_p)$] is the mean vertical (kinematic) flux density of sensible heat.

- The kinetic energy of the unresolved motion $u_i' = (u', v', w')$ is by definition $k = (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})/2$. In a horizontally-homogeneous layer the conservation equation for k, in a coordinate system aligned with the mean wind, is:

  $$\frac{\partial k}{\partial t} = -\,\overline{u'w'}\,\frac{\partial U}{\partial z} + \frac{g}{T_0}\,\overline{w'T'} - \epsilon + \frac{\partial}{\partial z}\,\overline{w'\left(\frac{p'}{\rho} + \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right)\right)}$$

  Here ϵ is the 'rate of dissipation of unresolved kinetic energy' and terms have their usual meaning.

- The heat flux budget equation in a horizontally-homogeneous layer is

  $$\frac{\partial \overline{w'T'}}{\partial t} = -\,\overline{w'^2}\,\frac{\partial \overline{T}}{\partial z} - \frac{\partial}{\partial z}\,\overline{w'w'T'} - \frac{1}{\rho_0}\,\overline{T'\frac{\partial p'}{\partial z}} + \frac{g}{T_0}\,\overline{T'^2} \tag{10}$$

  where ρ_0, T_0 are the mean density and (Kelvin) temperature of the layer. It is often assumed that at steady state this reduces to a balance between the first term on the r.h.s. (gradient production: a source term) and the $T'\,\partial p'/\partial z$ covariance term, also a source term.

Skeleton short answers

1. Regarding the "neutrally-stratified" atmospheric boundary-layer, any of these points would gain you a mark:

   - $Q_H = 0$, or alternatively $\overline{w'T'} = 0$
   - $\partial\overline{T}/\partial z - \gamma_d = 0$, where $\overline{T}$ is the **mean** temperature; alternatively $\partial\overline{\theta}/\partial z = 0$, where $\overline{\theta}$ is the **mean** potential temperature (since in principle the temperature fluctuations vanish, no penalty for failing to specify 'mean')
   - no buoyant production in TKE equation, so (at local equilibrium and steady state) $\epsilon = -\overline{u'w'}\,\partial U/\partial z$ (a half mark if you got part way to this result)
   - in the surface layer the MO scale $T_* = -Q_H/u_*$ for temperature fluctuations vanishes... no temperature fluctuations
   - in the surface layer the MO length scale $|L| = \infty$ ($z/L = 0$)
   - parcels displaced in the vertical experience no consequent buoyancy force
   - neutral ABL can be said to be "well-mixed"

2. Evaluate $T(y) = \sum_{i=1}^{20} \sum_{j=i}^{20} (-1)^{i+j} \sin(\pi y)$ for a bunch of values of y. What was tested here was your familiarity with nested loops, and the type of variable generally called an "accumulator" (here named "sum"). Marked by subtracting $\frac{1}{2}$ for each mistake. Common mistakes were: wrong lower limit for the j loop; failure to actually so the summation (as marked 'wrong' below) with result that computation yields only the contribution from $i = j = 20$.

```
   integer i,j
   real    pi,sum,y,x(10)
   pi=3.14159
   read x(k) for k=1,..10        (optional)
   do k=1,10
     y=x(k)
     sum=0.0
     do i=1,20
       do j=i,20
         sum=sum+((-1)**(i+j))*sin(pi*y)
c        sum=((-1)**(i+j))*sin(pi*y)           wrong!
       end do
```

```
    end do
    print*, y, sum              (optional)
  end do
  end
```

3. The $U\frac{\partial\phi}{\partial x} = \frac{\partial^2\phi}{\partial z^2}$ problem. Marked by assigning +1 per point below (to max of 5)

- to fully pose the problem (ie. render it definite and so solvable) we need 1 b/c on the x-axis and 2 on the z-axis
- on the z-axis, the problem is 'elliptic' ('jury' problem), this is a '2-way' axis
- however the x-axis is a '1-way' axis, and the solution $\phi_{i,j}$ at gridpoint (i,j) can be affected only by the upstream field $\phi_{i-1,k}$ (if we take the simplest discretization of the Laplacian term, the upstream influence is restricted to height range $j-1 \leq k \leq j+1$)
- An advection term $U\partial\phi/\partial x$ is best discretized using an 'upwind' difference (ie. if I want $\partial\phi/\partial x$ where I am standing with my back to the wind, I take the value of ϕ where I am minus the value behind my back, and divide by the distance)... this captures the '1-way' character of the transport process. From the physical perspective a boundary condition at $x = L$ would not work (it could not influence the solution at $x < L$). However since I did not clearly teach and give you notes on this point (inferiority of a downwind difference for advection) and because mathematically, by choosing a downwind difference for the advection term, one *could* obtain a solution even if one imposed the b/c at $x = L$, I accepted either argument.
- since $U =$ const both terms in the d.e. are transport terms, ie. can be written as the spatial derivative of a flux
- The simplest discretization for the Laplacian term $\left(\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}\right)/\Delta^2$ is going to zip the influence of the lower boundary condition upwards along the z (ie. j) axis by one gridpoint per iteration. So after 4 iteration the boundary influence is felt up to the 4^{th} internal gridpoint; however since there can be ambiguities about the counting here, I did not insist on 4

- Since this is (or can be formulated as) a marching problem on the x-axis, iteration is not actually necessary... but then the influence of the lower boundary will propagate up the j-axis by one gridpoint for each downstream step taken

4. Letting ϕ_i^n represent the solution at time $n\Delta t$ and location $i\Delta$, an explicit algorithm for $\frac{\partial \phi}{\partial t} = a\,\frac{\partial^4 \phi}{\partial x^4}$ would be

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{a}{\Delta^4}\,\left(\phi_{i+2}^n - 8\phi_{i+1}^n + 20\phi_i^n - 8\phi_{i-1}^n + \phi_{i-2}^n\right)$$

so in terms of neighbour coefficients we have (3 marks)

$$\begin{aligned} \phi_i^{n+1} = & + \gamma\,\phi_{i-2}^n \\ & - 8\gamma\,\phi_{i-1}^n \\ & + (1 + 20\gamma)\,\phi_i^n \\ & - 8\gamma\,\phi_{i+1}^n \\ & + \gamma\,\phi_{i+2}^n \end{aligned}$$

where $\gamma = a\Delta t/\Delta^4$. Because this algorithm requires values at gridpoints $i-2, i-1, i, i+1, i+2$ there would be 5 non-zero diagonals in the coefficient matrix of an implicit formulation (2 marks).

5. Interpreting/explaining the boundary flux parameterization $Q_{H0} = \rho c_p\, C_M\, C_H\, u_a\, (T_0 - T_a)$.

- formulation assumes the sensible heat flux density Q_{H0} is 'driven' by the (model's) surface-to-air temperature difference $(T_0 - T_a)$
- but we need to multiply by a velocity to get units of kinematic heat flux, and u_a (model's mean horizontal windspeed at $z = z_a$) is available from the model (other choices of a windspeed 'known to the model' would be possible, but would result in different coefficients)
- the factor ρc_p converts our kinematic heat flux density into the units [W m^{-2}]
- to get the (**dimensionless**) coefficients, we turn to Monin-Obukhov similarity theory (MOST), which is appropriate for a horizontally uniform surface layer

(since our model doesn't resolve the surface layer on a scale finer than its grid, the assumption of horizontal uniformity that is implicit in MOST is consistent with the modelling approach). We start by noting that according to MOST

$$\begin{aligned} \frac{k_v z}{u_*} \frac{\partial \overline{u}}{\partial z} &= \phi_m \left(\frac{z}{L} \right) \\ \frac{k_v z}{T_*} \frac{\partial \overline{T}}{\partial z} &= \phi_h \left(\frac{z}{L} \right) \end{aligned}$$

where $T_* = -Q_{H0}/(\rho c_p u_*)$ and ϕ_h is the universal MO function for heat (the first of these equations was given as data). Re-arrange the second of these formulae

$$\frac{\partial \overline{T}}{\partial z} = \frac{-Q_{H0}/(\rho c_p u_*)}{k_v z} \, \phi_h(z/L)$$

and upon integrating from the surface to $z = z_a$

$$T_a - T_0 = \frac{-Q_{H0}/(\rho c_p u_*)}{k_v} \int_0^{z_a} \phi_h(\frac{z}{L}) \, \frac{dz}{z}$$

we have a formula linking Q_{H0} to $(T_0 - T_a)$ that can be used to determine $C_M C_H$ (the coefficient C_M enters when we re-express u_* as a multiple of u_a)

6. Control volume discretization of $0 = \frac{d}{dz} \left(K(z) \, \frac{dC}{dz} \right) + \, q_1 \, \delta(z - h)$.

- We have a steady-state, 1-d diffusion problem. There is a source (strength q_1) of the property C inside the domain at $z = h$, and another (strength q_2) at the lower boundary $z = 0$ (since we have a prescribed flux at that boundary). The top boundary $z = L$ is either very remote from the sources, or (better interpretation) is a perfect sink
- define a uniform grid along the z axis with gridlength Δ
- index the gridpoints (j)
- sketch the grid giving definitions
- integrate the d.e. over the interval $z_s \leq x \leq z_n$ where z_s, z_n are the upper and lower boundaries to the control volume (or control layer) associated with gridpoint j. In view of the δ-function multiplying the source term, the result is

$$0 = \left[K(z) \, \frac{dC}{dz} \right]_{z_s}^{z_n} + \, q_1 \delta_{j j_h}$$

where δ_{jj_h} vanishes unless layer j encloses the source at $z = h$ (in which case it is unity)

- Introducing central differences we have

$$0 = K(z_n)\, \frac{C_{j+1} - C_j}{\Delta} - \; K(z_s)\, \frac{C_j - C_{j-1}}{\Delta} + \; q_1 \delta_{jj_h}$$

or rearranging

$$C_j \; \left[\frac{K_n}{\Delta} + \frac{K_s}{\Delta} \right] = \; C_{j-1}\, \frac{K_s}{\Delta} + \; C_{j+1}\, \frac{K_n}{\Delta} + \; q_1 \delta_{jj_h}$$