EAS471, "Atmospheric Modelling" <u>Exam</u> 19 April, 2007

<u>Professor</u>: J.D. Wilson <u>Time available</u>: 120 mins <u>Value</u>: 35%

Please answer in the exam booklet. Symbols have their usual meteorological interpretation (see details given at the back). *Note: Schematic answers added at back, 23 April, 2007.*

Multi-choice (14 x 1% = 14%)

1. $\tau_{ij} \equiv \overline{u'_i u'_j}$ is the Reynolds stress tensor, and $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the Kronecker delta. The quantity $\tau_{ij}\,\delta_{ij}/2$ expands to
   (a) $U\,\partial\overline{u'^2}/\partial x + V\,\partial\overline{v'^2}/\partial y + W\,\partial\overline{w'^2}/\partial z$
   (b) 0
   (c) 1
   (d) $-u_*^2$ (where u_* is the friction velocity)
   (e) the turbulent kinetic energy $k = (\sigma_u^2 + \sigma_v^2 + \sigma_w^2)/2$ ✓✓

2. In a purely gaseous atmosphere, longwave radiation can be treated ____ , while (with the exception of the ultraviolet and near infra-red spectral bands where absorption is significant) solar radiation can be treated ____
   (a) neglecting absorption and emission; neglecting scattering
   (b) neglecting scattering; neglecting absorption ✓✓
   (c) assuming an isotropic scattering function; neglecting extinction
   (d) assuming an isotropic scattering function; assuming isotropic Planck (black body) emission
   (e) assuming Rayleigh scattering; by assuming the photon mean free path is zero

3. Under the assumption of a "plane parallel atmosphere," the intensity of diffuse solar radiation $I_\lambda(p, \theta, \phi)$ [$\mathrm{J\,s^{-1}\,m^{-2}\,\mu m^{-1}\,steradian^{-1}}$] is ____
   (a) independent of time
   (b) independent of wavelength λ
   (c) independent of height (indexed above by pressure, p)
   (d) independent of the elevation angle θ measured away from the zenith
   (e) independent of the azimuth angle ϕ measured away from due north ✓✓

4. The solid angle $d\omega$ subtended at the origin by an elementary area dA *of a spherical surface* at a distance r from the origin is ____ (note added 23 April 2007: qualification in italics ought to have been present)

   (a) $d\omega\, dA$

   (b) $r\, dA$

   (c) $r^2\, d\omega/A$

   (d) dA/r^2 ✓✓

   (e) $r\, d\omega$

5. If an atmosphere is optically ____ at wavelength λ, then radiant energy transport at that wavelength can be treated as a ____ process controlled by the spatial gradient in the ____

   (a) thin; convection; temperature

   (b) thick; diffusion; Planck emission/absorption function ✓✓

   (c) thin; diffusion; Planck emission/absorption function

   (d) thick; scattering; photon mean free path

   (e) transparent; diffusion; wavelength

6. According to the Lax Equivalence Theorem, "If a difference equation is consistent with the differential equation it represents, then stability is the necessary and sufficient condition for convergence." Here the technical meaning of "consistent" is that

   (a) truncation error must vanish in the limit of vanishing grid interval(s) ✓✓

   (b) truncation error must vanish in the limit of infinite grid interval(s)

   (c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s)

   (d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

7. According to the Lax Equivalence Theorem, "If a difference equation is consistent with the differential equation it represents, then stability is the necessary and sufficient condition for convergence." Here the technical meaning of "convergence" is that

   (a) truncation error must vanish in the limit of vanishing grid interval(s)

   (b) truncation error must vanish in the limit of infinite grid interval(s)

   (c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s) ✓✓

   (d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

8. Which of the following is *not* a limitation/approximation/restriction/assumption of the quasi-geostrophic system of equations?

   (a) restricted to extra-tropical region

   (b) latitudinal variation of Coriolis parameter approximated as linear ($f = f_0 + \beta y$)

   (c) advection approximated by substituting the Geostrophic wind for the true wind

   (d) horizontal divergence D_p of the *true* wind assumed to vanish ✓✓

   (e) vertical advection of vorticity neglected

9. The von Neumann stability analysis of the heat equation $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$ results in a criterion on the ____ number

   (a) Diffusion ✓✓

   (b) Reynolds

   (c) Richardson

   (d) Courant

   (e) Euler

10. Under certain simplifying assumptions and in the usual notation, the rate of shear production of turbulent kinetic energy can be written $-\overline{u'w'}\,\partial U/\partial z$, while buoyant production is $(g/T_0)\,\overline{w'T'}$. The 'flux Richardson number' is defined to be ____

    (a) minus the ratio of buoyant over shear production of turbulent kinetic energy ✓✓

    (b) z/L (where L is the Obukhov length)

    (c) $\overline{w'T'}/L$

    (d) $\overline{w'T'}/T_0$ (where T_0 is a bulk Kelvin reference temperature)

    (e) $-\overline{w'T'}/\overline{u'w'}$

11. Suppose the Probability Density Function (PDF) of a certain random variable x defined on $-\infty \leq x \leq \infty$ is

$$f(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & x < a \\ 0 & x > b \end{cases} \tag{1}$$

    Then the mean square value of x is $\mathrm{E}[x^2] = \int_{-\infty}^{\infty} x^2 f(x)dx =$ ____

    (a) 0

    (b) 1

    (c) -1

    (d) $(1/3)\,(b^3 - a^3)\,(b-a)^{-1}$ ✓✓

    (e) $a\,b/3$

12. Non-linear computational instability (NLCI) of a finite difference scheme is a consequence of which factor(s)?

   (a) false diffusion, use of low-order computational molecules
   (b) aliasing, truncation in spectral space
   (c) failure to reformulate advection terms in flux form
   (d) wave-wave interaction, aliasing ✓✓
   (e) truncation error

13. Suppose $L\left[\phi(x)\right] = 0$ (where $L[\ ]$ is a differential operator) is to be solved on $a \leq x \leq b$ using the Bubnov-Galerkin method with basis functions $\theta_j(x)$, viz. $\phi(x) = \sum_j \, a_j \, \theta_j(x)$. The (constant) expansion coefficients a_j are optimized by requiring that ____

   (a) $a_i a_j = \delta_{ij}$
   (b) the residual $e(x) = \; L\left[\sum_j \; a_j \; \theta_j(x)\right]$ must vanish
   (c) the residual $e(x)$ must be minimal
   (d) the covariance (over $a \leq x \leq b$) of the residual $e(x)$ with each basis function $\theta_j(x)$ must vanish ✓✓
   (e) the residual $e(x)$ must be maximally correlated with each $\theta_j(x)$ over $a \leq x \leq b$

14. Gridpoint computations for the influence of unresolved scales of motion in the ABL on the resolved absolute humidity ρ_v will involve the equation

$$\left(\frac{\partial \rho_v}{\partial t}\right)^{phys} \;=\; - \;[.]\; \overline{w'\rho_v'} \tag{2}$$

   The missing operator "[.]" is ____

   (a) $U\,\partial/\partial x + \; V\,\partial/\partial y$ (U, V the resolved horizontal velocity components)
   (b) $W\,\partial/\partial z$ (W the resolved vertical velocity)
   (c) $\partial/\partial x$
   (d) $\partial/\partial z$ ✓✓
   (e) $K\,\partial/\partial z$ (K the eddy diffusivity)

Short answer: $\quad$ 3 x 7% = 21%

Answer any **three** questions from this section, writing a (**maximum**) of one single-sided page per question. Additional pages used for calculations/working/preparation will not be marked.

1. Assuming a hydrostatic atmosphere, the (vertical) vorticity equation is

$$\begin{aligned}\frac{\partial \zeta}{\partial t} \; + \; \vec{V} \cdot \nabla_H \left(\zeta + f \right) \; + \omega \frac{\partial \zeta}{\partial p} \\ = \; - \left(\zeta + f \right) \, \nabla_H \cdot \vec{V} \; + \hat{k} \cdot \left(\frac{\partial \vec{V}}{\partial p} \times \nabla_H \omega \right)\end{aligned} \tag{3}$$

where $\vec{V}$ is the horizontal wind vector and $\omega \, [\mathrm{Pa\,s^{-1}}]$ the vertical velocity. Explain the further limitations/assumptions/substitutions and/or simplifications that lead to the quasi-geostrophic vorticity equation

$$\frac{\partial \zeta}{\partial t} = - U_g \frac{\partial \zeta}{\partial x} - V_g \frac{\partial \zeta}{\partial y} - V_g \beta + f_0 \frac{\partial \omega}{\partial p} \tag{4}$$

where $\vec{V_g}$ is the Geostrophic wind vector defined at a reference latitude ϕ_0, f_0 is the Coriolis parameter at that latitude, and $\beta = (\partial f / \partial y)_{\phi_0}$ its northerly gradient.

2. Conservation of the mass of a pollutant in the atmosphere is expressed

$$\frac{\partial c}{\partial t} = - \nabla \cdot \vec{F_c} + q \tag{5}$$

where $c = c(x, y, z, t)$ is the instantaneous concentration, $\vec{F_c}$ the instantaneous vector flux density of c, and q the volumetric source/sink density. In the case of turbulent dispersion in a horizontally uniform atmospheric surface layer, the mean concentration C due to a continuous line source is often modelled by the advection-diffusion equation

$$U \frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left(K \frac{\partial C}{\partial z} \right) \tag{6}$$

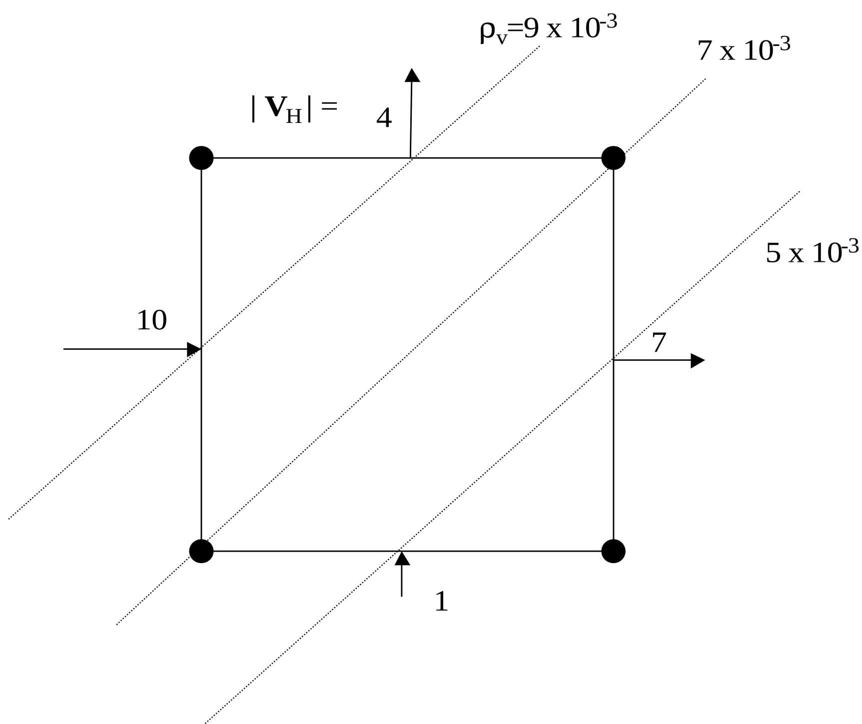
where $U = U(z)$ is the mean velocity and $K = K(z)$ is the eddy diffusivity. List and justify the simplifications and restrictions and assumptions underlying this model (most safely done by deriving it from eqn. 5).

3. Using the control volume method and assuming constant but unequal gridlengths $(\Delta x, \Delta y)$, discretize the differential equation

$$0 = \nabla^2 T + h(x, y) \tag{7}$$

governing $T = T(x, y)$ on the domain $0 \le x \le X$, $0 \le y \le L$. State the neighbour coefficients A^N, A^S, A^E, A^W, A^C (North, South, East, West and Centre) for an arbitrary internal control volume (ie. a gridpoint not lying on a boundary), and give a diagram of your grid that clarifies your procedure. Give also a physical interpretation of the governing equation, ie. explain speculatively what physical problem it models, and comment on the significance of the boundary condition (not specified above).

4. In the accompanying diagram, the separation between the gridpoints is 200 km. Arrows with affixed numbers give the direction and magnitude [$\mathrm{m\,s^{-1}}$] of the horizontal wind vector $\vec{V}_H$ at each interface, and the diagonals are contours of absolute humidity ρ_v [$\mathrm{kg\,m^{-3}}$]. For this cell calculate: the Laplacian of the humidity, $\nabla^2 \rho_v$; the velocity divergence, $\nabla_H \cdot \vec{V}_H$; and the rate of moisture accession by horizontal advection, $\nabla_H \cdot \left(\rho_v \vec{V}_H\right)$.



5. In a particular NWP model, vertical fluxes carried by the unresolved scales of motion are modelled using an eddy diffusivity (word should have been '*viscosity*') $K = a\lambda\sqrt{k}$, where k is the "turbulent kinetic energy" lying in the unresolved scales and λ is a length scale. Assuming a neutral surface layer near ground in which

$$\begin{aligned} U &= \frac{u_*}{k_v} \ln\left(\frac{z}{z_0}\right) \\ k &= 5u_*^2 \\ \lambda &= k_v z \end{aligned} \tag{8}$$

(where U is the mean windspeed, u_* is the friction velocity, z_0 is the roughness length, and $k_v = 0.4$ is the von Karman constant), determine the value of the proportionality constant "a" in the eddy diffusivity (should have been 'viscosity') that ensures exact satisfaction of the steady-state turbulent kinetic energy balance equation:

$$\frac{\partial k}{\partial t} = 0 = K\left(\frac{\partial U}{\partial z}\right)^2 + \frac{\partial}{\partial z}\left(K\,\frac{\partial k}{\partial z}\right) - \frac{k^{3/2}}{\lambda} \tag{9}$$

Data

- Cartesian velocity components (u, v, w) along the Cartesian coordinate directions (x, y, z)
- Reynolds decomposition $a = A + a'$, where A is the mean and a' the fluctuation
- Obukhov length $L = -\,u_*^3\,T_0\,\left(k_v\,g\,\overline{w'T'}\right)^{-1}$

Skeleton answers to the short answer questions

1. This question was somewhat given away, by multichoice question #8. It was marked by subtracting 1 mark for any essential step missed. The steps required to obtain the QG vorticity equation were

   - restrict the latitude to the extra-tropics
   - approximate latitudinal variation of Coriolis parameter as linear, $f(y) = f_0 + \beta y$ ('beta-plane approximation')
   - evaluate horizontal advection of vorticity using the Geostrophic (as opposed to true) windspeed, the latter evaluated using $f = f_0$
   - neglect vertical advection of vorticity
   - neglect the term involving $\partial \vec{V} / \partial p$

   Extra clarification that *could* have been given included:

   - that the term $V_g \, \beta$ originates from the term in northward advection of absolute vorticity, viz. from $V_g \, \partial(\zeta + f_0 + \beta y)/\partial y$

   - that the term $f_0 \, \partial\omega/\partial p$ stems from $(\zeta + f) \; \nabla_H \cdot \vec{V}$ by using the continuity equation $\nabla_H \cdot \vec{V} \equiv -\partial\omega/\partial p$, and adopting the approximation $\zeta + f \approx f_0$

2. Question was marked by subtracting 1/2 mark for every essential logical step missing. The necessary steps are:

   - Identify the flux, $\vec{F}_c = \vec{u} c - \kappa \nabla c$ where κ is the molecular diffusivity of c in air. For the subsequent steps it is easiest to use the tensor notation, $F_{ic} = u_i c - \kappa \partial c / \partial x_i$
   - neglect molecular diffusion (drop terms in κ... can be done later).
   - neglect any internal production/destruction, i.e. $q = 0$
   - Reynolds average, $c = C + c'$ (etc.) where $C \equiv \overline{c}$ is the mean.
   - use an explicitly defined terminology that distinguishes c from C, u from U, etc.

- upon Reynolds averaging the instantaneous equation transforms to

$$\begin{aligned}\frac{\partial C}{\partial t} &= -\frac{\partial}{\partial x_i}\left(\overline{u_i\, c}\right)\\ &= -\frac{\partial}{\partial x_i}\left(U_i\, C + \overline{u_i' c'}\right)\end{aligned}$$

- assume the concentration field is in steady state

$$0 = -\frac{\partial}{\partial x_i}\left(U_i\, C + \overline{u_i' c'}\right)$$

- assume line source lies parallel to the y axis, so that mean concentration field is invariant along y

$$0 = -\frac{\partial}{\partial x}\left(U\, C + \overline{u' c'}\right) - \frac{\partial}{\partial z}\left(W\, C + \overline{w' c'}\right)$$

(n.b. that this does *not* imply $\partial F_{yc}/\partial y = 0$, for the latter is the divergence of the instantaneous flux, a turbulent quantity)

- in the hhASL, $W = 0$ and $U = U(z)$, so

$$0 = -U\frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\overline{u' c'} - \frac{\partial}{\partial z}\overline{w' c'}$$

- neglect alongwind transport by the fluctuations, relative to transport by the mean flow.

- Assume the vertical eddy flux can be modelled as a diffusion process with eddy diffusivity K, viz

$$\overline{w' c'} = -K\frac{\partial C}{\partial z}$$

Explicitly distinguish this eddy diffusivity from the neglected molecular diffusivity κ

- in cumulation, these steps (whose ordering does matter) yield the approximation

$$U\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(K\frac{\partial C}{\partial z}\right)$$

3. Having given a diagram defining the indices, the general gridpoint (I, J), and the faces (e, w, n, s) of the control volume it lies within, perform an integration

$$0 = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dx\, dy + \overline{h}\, \Delta x\, \Delta y$$

(where $\overline{h}$ is the mean value for the control volume in question). This simplifies to

$$0 = \Delta y \left[\frac{\partial T}{\partial x} \right]_w^e + \Delta x \left[\frac{\partial T}{\partial y} \right]_s^n + \overline{h}\, \Delta x\, \Delta y$$

where the face 'e' is defined by $x = x_2$ (etc). Now taking finite differences we have

$$0 = \Delta y \left[\frac{T_{I+1,J} - T_{I,J}}{\Delta x} - \frac{T_{I,J} - T_{I-1,J}}{\Delta x} \right] + \Delta x \left[\frac{T_{I,J+1} - T_{I,J}}{\partial y} - \frac{T_{I,J} - T_{I,J-1}}{\partial y} \right] + \overline{h}_{I,J}\, \Delta x\, \Delta y$$

This can be rearranged to the form:

$$0 = A^N\, T_{I,J+1} + A^S\, T_{I,J-1} + A^E\, T_{I+1,J} + A^W\, T_{I-1,J} + A^C\, T_{I,J}\, \overline{h}_{I,J}\, \Delta x\, \Delta y$$

where

$$\begin{aligned} A^N \equiv A^S &= \Delta x / \Delta y \\ A^E \equiv A^W &= \Delta y / \Delta x \\ A^C &= -\,2\,(\Delta x/\Delta y + \Delta y/\Delta x) \end{aligned}$$

(Note that if $\Delta x = \Delta y$ this simplifies considerably.)

As to the origin of the equation, one could plausibly regard it as emerging from a balance of the form (see also short answer question #1)

$$\frac{\partial c}{\partial t} = 0 - \nabla \cdot \vec{F}_T + h$$

where

$$\vec{F}_T = -\kappa\, \nabla T$$

Thus we have a steady state diffusion problem in the plane, with a distributed sink/source $h(x, y)$ (equivalently, we can consider it a conduction problem, in which case T specifically is a temperature). Note especially that there is no advection process for T in this problem. Either the diffusivity (equivalently, conductivity) $\kappa = 1$, or, its value has been absorbed into the definition of h. The curvature of the field of $T(x, y)$ is constrained to equal $-h$. This is an elliptic (or 'jury') problem, and the solution for $T(x, y)$ is controlled by the (unspecified) boundary values.

If the explicit integration step was not shown, and you had simply plugged computational molecules for $\partial^2 T/\partial x^2, \partial^2 T/\partial y^2$ into the differential equation, then you had not used a 'control volume method' and I imposed a penalty 1 mark loss.

4. This involved straightforward application of the computational molecules for the first derivative, and for the curvature (ie. second derivative). Don't forget $\nabla^2()$ has *two* contributions, ie. curvature on x-axis plus curvature on y-axis... in this case, both zero. Give the units, even if the numerical value is zero.

$$\begin{aligned}
\nabla^2 \rho_v &= 0 \ [\mathrm{kg\,m^{-5}}] \\
\nabla_H \cdot \vec{V}_H &= 0 \ [\mathrm{s^{-1}}] \\
\nabla_H \cdot \left(\rho_v \vec{V}_H\right) &= -1.2 \times 10^{-7} \ [\mathrm{kg\,m^{-3}\,s^{-1}}]
\end{aligned}$$

5. It is crucial to recognize that the surface-layer velocity scale u_* is to be regarded as a constant, in which case $\partial k/\partial z = 0$ and the transport term in the TKE balance vanishes.

To evaluate the shear production term, differentiate the given formula for $U(z)$, viz.

$$\begin{aligned}
\frac{\partial U}{\partial z} &= \frac{u_*}{k_v} \frac{\partial}{\partial z} \ln\left(\frac{z}{z_0}\right) \\
&= \frac{u_*}{k_v} \frac{1}{z_0} \frac{\partial}{\partial z/z_0} \ln\left(\frac{z}{z_0}\right) \\
&= \frac{u_*}{k_v} \frac{1}{z_0} \frac{1}{z/z_0} \\
&= \frac{u_*}{k_v\, z}
\end{aligned} \tag{10}$$

Straightforward substitution gives $a = 5$.

Though it does not bear on the marking of the question, the question was subtly wrong in the way the dissipation term was set up, and ought to have been framed this way:

Revised framing of short answer question #5

In a particular NWP model, vertical fluxes carried by the unresolved scales of motion are modelled using an eddy viscosity $K = \lambda\sqrt{a\,k}$, where k is the "turbulent kinetic energy" lying in the unresolved scales and λ is a length scale. This eddy viscosity is to be 'calibrated' for a reference flow, specifically a neutral surface layer near ground in which

$$\begin{aligned} U(z) &= \frac{u_*}{k_v} \ln\left(\frac{z}{z_0}\right) \\ k &= 5u_*^2 \\ \lambda(z) &= k_v z \end{aligned} \tag{11}$$

where U is the mean windspeed (the von Karman constant k_v, the friction velocity u_*, and the roughness length z_0 are all to be regarded as constants). Determine the value of the proportionality constant "a" in the eddy viscosity that ensures exact satisfaction (in the given reference flow) of the steady-state turbulent kinetic energy balance equation:

$$\frac{\partial k}{\partial t} = 0 = K\left(\frac{\partial U}{\partial z}\right)^2 + \frac{\partial}{\partial z}\left(K\,\frac{\partial k}{\partial z}\right) - \frac{(a\,k)^{3/2}}{\lambda} \tag{12}$$

The answer to this corrected statement would be: $a = 1/5$.