EAS471, "Atmospheric Modelling" Exam 20 April, 2009

Professor: J.D. Wilson Time available: 120 mins Value: 30%

Please answer in the exam booklet. Symbols have their usual meteorological interpretation. Some data are given at the back. *Schematic answers to the short answer questions have been added at the back (April 21, 2009)*

Multi-choice (20 x $\frac{1}{2}$% = 10%)

1. When the control volume method is applied as the first step in discretizing a conservation equation such as
$$\frac{\partial \theta}{\partial t} = -\nabla \cdot \vec{f} + Q$$
the transport term ____
   (a) vanishes
   (b) must be Reynolds-averaged
   (c) can be neglected relative to molecular transport
   (d) finds expression in the net flux into the control volume (cell) across its walls ✓✓
   (e) equates to the mean value of Q within the cell's volume $\Delta x \Delta y \Delta z$

2. In Cartesian coordinates (x, y, z) the quantity $\nabla \cdot \nabla \phi$ equates to ____
   (a) 0
   (b) $\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial z}$
   (c) $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}$
   (d) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ ✓✓
   (e) 1

3. Suppose $\phi = \phi(x, t)$ is governed on $-\infty \leq x \leq \infty$ by the one-dimensional, linear advection equation
$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = 0$$
(where c is a constant). Then if the initial field is $\phi(x, 0) = \phi_0(x) = \sin(2\pi x/L)$, the general solution is ____
   (a) $\phi(x, t) = \sin(2\pi(x - c\,t)/L)$ ✓✓
   (b) $\phi(x, t) = \sin(2\pi x/L) - c\,t/L$
   (c) $\phi(x, t) = \phi_0(x)$
   (d) $\phi(x, t) = \phi_0(x/L + c\,t)$
   (e) $\phi(x, t) = \phi_0(x/L - c)$

4. Suppose that in a certain spectral NWP model the variation of ϕ with coordinate x was represented as

$$\phi(x) = \sum_{n=-\infty}^{\infty} a_n \, e^{j\,n\,k_0\,x} \,,$$

where $j = \sqrt{-1}$, k_0 is some fundamental wavenumber, and the a_n are expansion coefficients that are independent of x. Then wherever needed, $\partial\phi/\partial x$ would be represented as ____

(a) $[\phi(x + \Delta x) - \phi(x - \Delta x)]/(2\Delta x)$

(b) $\sum_{n=-\infty}^{\infty} a_n \left[e^{j\,n\,k_0\,(x+\Delta x)} - e^{j\,n\,k_0\,(x-\Delta x)}\right]/(2\Delta x)$

(c) $\sum_{n=-\infty}^{\infty} j\,n\,k_0\,a_n\,e^{-j\,n\,k_0\,x}$

(d) $\sum_{n=-\infty}^{\infty} j\,n\,k_0\,a_n\,e^{j\,n\,k_0\,x}$ ✓✓

(e) $\sum_{n=-\infty}^{\infty} \frac{\partial a_n}{\partial x}\,e^{j\,n\,k_0\,x}$

5. In the context of the quasi-geostrophic model of mid-latitude meteorological dynamics expressed in the (x, z, p, t) coordinate system, the height Z of a constant pressure surface is related to the vertical component of relative vorticity ζ through a Poisson equation

$$\nabla^2 Z = \frac{f_0}{g}\,\zeta \,,$$

where f_0 is the Coriolis parameter at a fixed central latitude and g is gravitational acceleration. The operator ∇ is ____

(a) $\nabla = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$

(b) $\nabla = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y}$ ✓✓

(c) $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

(d) $U_g\,\frac{\partial}{\partial x} + V_g\,\frac{\partial}{\partial y}$, where $\vec{U}_g$ is the geostrophic wind

(e) $\frac{\partial}{\partial t} + U_g\,\frac{\partial}{\partial x} + V_g\,\frac{\partial}{\partial y}$, where $\vec{U}_g$ is the geostrophic wind

6. The problem of solving the above Poisson equation for the height field Z given the forcing field $\zeta(x, y)$ is classified as a "____ " problem, because the (x, y) axes are ____

(a) marching; one-way

(b) marching; two-way

(c) jury; one-way

(d) jury; two-way ✓✓

(e) non-linear; periodic

7. An eddy diffusion closure for the unresolved (kinematic) vertical heat flux density $\overline{w'\theta'}$ reads

$$\overline{w'\theta'} = -\, K_h \, \frac{\partial \overline{\theta}}{\partial z}$$

where θ is the potential temperature. Here K_h is the ____

(a) molecular conductivity $[\mathrm{W\,m^{-1}\,K^{-1}}]$
(b) molecular thermal diffusivity $[\mathrm{m^2\,s^{-1}}]$
(c) eddy diffusivity for heat $[\mathrm{m^2\,s^{-1}}]$ ✓✓
(d) eddy viscosity $[\mathrm{m^2\,s^{-1}}]$
(e) rate of dissipation of turbulent kinetic energy $[\mathrm{m^2\,s^{-3}}]$

8. Suppose the mean windspeeds at two heights $(z_1,\, z_2)$=(1.2, 2.4) m in a neutrally-stratified atmospheric surface layer were $U_1 = 3\ \mathrm{m\,s^{-1}}$ and $U_2 = 3.5\ \mathrm{m\,s^{-1}}$. Based on the observed mean wind shear, the friction velocity was ____ $\mathrm{m\,s^{-1}}$

(a) -0.72
(b) 0.29 ✓✓
(c) 0.72
(d) 1.10
(e) 2.74

9. Taylor gave a Lagrangian theory of dispersion in homogeneous, stationary turbulence, which we here apply to the spread of particles along the vertical (z) axis. To reconcile Taylor's theory with the eddy diffusion treatment one finds the eddy diffusivity must be a function of the travel time (t) since the particles were released into the flow, viz.

$$K = \sigma_w^2 \int_0^t R(\tau)\, d\tau \qquad (1)$$

where σ_w^2 is the variance of the vertical velocity and $R(\tau)$ is the Lagrangian velocity autocorrelation function. If $T_L = \int_0^\infty R(\tau)\, d\tau$ is the Lagrangian integral timescale, then the "far field eddy diffusivity" is given by ____

(a) $\sigma_w\, T_L$
(b) $\sigma_w\, t$
(c) $\sigma_w^2\, T_L$ ✓✓
(d) $\sigma_w^2\, t$
(e) $\sqrt{2\sigma_w^2\, T_L\, t}$

10. If advection by the resolved wind field were the sole mechanism causing evolution of the temperature at a point in the atmosphere (or ocean), then the sign and magnitude of the local tendency in (resolved) temperature $\partial T/\partial t$ would be given by ____

(a) $U_j\, \partial T/\partial x_j$

(b) $-\, U_j\, \partial T/\partial x_j$ ✓✓(same as $-\vec{U} \cdot \nabla T$)

(c) $-\, \vec{U} \times \nabla T$

(d) $\vec{U} \cdot \nabla T$

(e) $-\, \partial \overline{w'T'}/\partial z$

where $\vec{U} \equiv U_j$ is the resolved wind vector and $\overline{w'T'}$ is the unresolved vertical heat flux density.

11. In the context of the quasi-geostrophic model the geostrophic wind is conveniently related

$$\vec{U}_g = \hat{k} \times \nabla \psi$$

to a streamfunction $\psi = gZ/f_0$ (where Z is the height of a constant pressure surface and f_0 the Coriolis parameter at a reference latitude). The west-east component U_g (directed along $\hat{\imath}$) is given by ____

(a) $\partial^2 \psi/\partial x^2$

(b) $-\partial \psi/\partial x$

(c) $\partial \psi/\partial y$

(d) $-\partial \psi/\partial y$ ✓✓

(e) $\partial \psi/\partial y - \partial \psi/\partial x$

12. The "curvature" of a scalar field $\phi(x, y, z)$ is given by (or measured by) application of the ____ operator

(a) ∇

(b) $\nabla \cdot \nabla$ ✓✓

(c) $\hat{k} \cdot \nabla$

(d) $\hat{k}\ \times$

(e) $\hat{k}\ \times \nabla$

13. When we use the Reynolds averaging rules to formally separate an atmospheric or oceanic field a into its resolved ($\overline{a}$) and unresolved (a') components, the expected value of the unresolved part is ____

(a) the standard deviation $\sigma_a \equiv \sqrt{\overline{a'^2}}$

(b) a'

(c) $a - \overline{a}$

(d) $\overline{a'} = 0$ ✓✓

(e) $\overline{a'} \ll \overline{a}$

14. Suppose one performed many independent measurements of the vertical displacement $Z(t)$ at time t of a tracer particle (fluid element) from its initial state $Z(0) = 0$, in a regime of homogeneous turbulence (assume there is no mean flow, so that the mean value $\overline{Z}(t)$ is zero). According to the "diffusion" paradigm (equivalent to the "drunkard's walk" description) one should find that $\overline{Z^2}(t)$, the mean square value of $Z(t)$ defined over the ensemble of independent trials, is proportional to ____

   (a) $\sqrt{t}$ ✓✓
   (b) t
   (c) $t^{2/3}$
   (d) t^2
   (e) $\exp(t)$

15. The sheltering effect of a long thin porous windbreak can be computed by the Reynolds-averaged Navier-Stokes equations, in which it is necessary to parameterize the interaction of the flow with the windbreak (otherwise one would be compelled to represent the hopelessly intricate geometry of the barrier and the microscopic details of flow in and about it). The salient effect is a loss of momentum to the barrier, and, assuming the barrier (height H) is located at $x = 0$ and oriented along the crosswind (y) axis, this drag on the wind may be represented by a localized source in the mean streamwise (x) momentum equation

$$U \frac{\partial U}{\partial x} + W \frac{\partial U}{\partial z} = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'w'}}{\partial z} - k_r \, U^2 \, \delta(x-0) \, s(z-H)$$

(k_r is a dimensionless "resistance coefficient" of the barrier; the localizing step function $s(z - H)$ is unity for $z \leq H$ and zero otherwise). The primary effect of the momentum sink is to induce a streamwise pressure gradient $\partial P/\partial x$ that is ____ of the barrier

   (a) adverse (decelerating) both upwind and downwind ✓✓
   (b) favourable (accelerating) both upwind and downwind
   (c) adverse upwind and favourable downwind
   (d) favourable upwind and adverse downwind
   (e) infinite "at" the barrier ($x = 0,\ z \leq H$) and zero upwind or downwind

16. According to the Lax Equivalence Theorem, "If a difference equation is *consistent* with the differential equation it represents, then stability is the necessary and sufficient condition for convergence." Here the technical meaning of "consistent" is that

   (a) truncation error must vanish in the limit of vanishing grid interval(s) ✓✓
   (b) truncation error must vanish in the limit of infinite grid interval(s)
   (c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s)
   (d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

17. According to the Lax Equivalence Theorem, "If a difference equation is consistent with the differential equation it represents, then stability is the necessary and sufficient condition for convergence." Here the technical meaning of "convergence" is that

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(b) truncation error must vanish in the limit of infinite grid interval(s)

(c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s) ✓✓

(d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

18. A von Neumann stability analysis of the heat equation $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$ discretized using a forward difference in time (timestep Δt) and a centred difference in space (gridlength Δx) results in a stability criterion that the Diffusion number ____ should not exceed 1/2

(a) $\kappa \, \Delta t/\Delta x^2$ ✓✓

(b) $\kappa \, \Delta x/\Delta t$

(c) $\Delta x \Delta t/\kappa$

(d) $\Delta x^2/(\kappa \, \Delta t)$

(e) $T^* = \kappa \, \Delta t^{-1}$

19. Suppose the Probability Density Function (PDF) of a certain random variable x defined on $0 \leq x \leq 2$ is $f(x) = x/2$. Any function of x, say $G = G(x)$, has expected value

$$\mathrm{E}[G(x)] = \int_0^2 G(x) \, f(x) dx \, ,$$

where (for example) $\mathrm{E}[x]$ is the mean $\overline{x}$. Accordingly the mean, mean square, and variance of x are ____

(a) 4/3, 2, 2/9 ✓✓

(b) $x/3$, $2x$, 0

(c) 4/3, 16/9, 4/3

(d) 2/3, 4/3, $2/\sqrt{3}$

(e) you own answer (if not given above):

20. Non-linear computational instability (NLCI) of a finite difference scheme is a consequence of which factor(s)?

(a) use of low-order computational molecules

(b) aliasing, truncation in spectral space

(c) failure to reformulate advection terms in flux form

(d) wave-wave interaction, aliasing ✓✓

(e) truncation error

Short answer: $\quad$ 2 x 10% = 20%

Answer any **two** questions from this section.

1. The Reynolds-averaged u-momentum equation, under the Boussinesq hypothesis, is:

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\,\frac{\partial \overline{u}}{\partial x} + \overline{v}\,\frac{\partial \overline{u}}{\partial y} + \overline{w}\,\frac{\partial \overline{u}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z} + \nu\nabla^2\overline{u}$$

   where f is the Coriolis parameter, ρ_0 is layer mean density, and $\overline{p}$ is the mean pressure deviation from a hydrostatic and adiabatic reference state. Deduce the acceleration due to the "physics" $(\partial\overline{u}/\partial t)_{phys}$ and explain the underlying physics of "friction" in a horizontally-uniform boundary layer. Explain how friction is typically represented in a weather or ocean model.

2. Determine the 4×4 tridiagonal coefficient matrix **M** and the right hand side **B** in a matrix expression of form $\mathbf{M}\,\mathbf{\Theta} = \mathbf{B}$ for the numerical solution of

$$\frac{\partial^2\theta}{\partial z^2} = 2$$

   on $0 \le z \le 2$, subject to $\theta(0) = \theta(1) = 0$. Set up your solution with four, equi-spaced gridpoints indexed $J = (1,2,3,4)$ positioned at $z_J = (0, 1/3, 2/3, 1)$. At internal gridpoints (J=2,3) the curvature is to be represented as

$$\frac{\partial^2\theta}{\partial z^2} = \frac{\theta_{J+1} + \theta_{J-1} - 2\,\theta_J}{\Delta z^2}\ .$$

   You need not invert **M**, nor obtain the (numerical) solution vector $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ since the continuous analytical solution $\theta(z)$ is accessible by elementary Calculus — provide the latter.

3. In Assignments (3,4) you simulated the vertical profile of crosswind-integrated concentration ("C") one hundred metres downstream from a point source of tracer gas in a neutral surface layer (specifically, Project Prairie Grass Run 57, for which known values of the friction velocity u_* and roughness length z_0 were given). Consistent (and, as judged by the observations, rather useful) results were obtained using two different methods, namely

   - by solving an *advection-diffusion equation* (Assignment 2), viz.

$$U\,\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(K\,\frac{\partial C}{\partial z}\right)$$

     where the eddy diffusivity $K = (k_v/S_c)\,u_*\,z$ and the mean wind $U = (u_*/k_v)\,\ln(z/z_0)$. The von Karman constant $k_v = 0.4$, while the Schmidt number S_c was a free constant.

- by performing a *Lagrangian stochastic trajectory simulation* in which you computed mean residence time within detector "volumes" of particles whose trajectories from the source you computed by assuming the horizontal velocity was U (as given above), and the vertical velocity was stochastic, with standard deviation $\sigma_w = b\, u_*$ and timescale $T_L = 2\sigma_w^2/(C_0\,\epsilon)$. In your simulation the TKE dissipation rate was specified $\epsilon = u_*^3/(k_v z)$, while $b = 1.3$ and C_0 was a free constant.

Based on your understanding of similarities and differences between these two approaches, explain why one might have *expected* these two methods to give consistent results. In doing so, you should exploit Taylor's result for the effective far field eddy diffusivity $\sigma_w^2\, T_L$ implied by the Lagrangian model — this provides a connection between the two free constants, S_c and C_0.

Data

- $(\hat{\imath}, \hat{\jmath}, \hat{k})$ are unit vectors along the Cartesian coordinate directions (x, y, z), and (u, v, w) are the corresponding Cartesian velocity components

- Dimensionless mean wind shear in the horizontally-uniform atmospheric surface layer

$$\frac{k_v z}{u_*}\frac{\partial \overline{u}}{\partial z} \equiv \frac{k_v}{u_*}\frac{\partial \overline{u}}{\partial \ln(z)} = \phi_m\left(\frac{z}{L}\right) .$$

  The universal Monin-Obukhov function on the r.h.s. has the limit $\phi_m(0) = 1$ in neutral stratification, ie. at $|z/L| = 0$. Similar expressions apply for gradients in mean temperature, humidity, etc. The von Karman constant $k_v = 0.4$, u_* is the friction velocity, and L is the Obukhov length

- The Obukhov length

$$L = -\frac{u_*^3\, T_0}{k_v\, g\, \overline{w'T'}}$$

  where T_0 [K] is the mean temperature of the layer

Schematic answers

1. The terms that are conventionally grouped together as representing "the acceleration due to the physics" are

$$\left(\frac{\partial \overline{u}}{\partial t}\right)_{phys} = -\frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z}$$

and in a horizontally-uniform boundary-layer (hhBL) by definition this reduces to

$$\left(\frac{\partial \overline{u}}{\partial t}\right)_{phys} = -\frac{\partial \overline{u'w'}}{\partial z} \ .$$

The Reynolds stress $\overline{u'w'}$ is (physically) the covariance of unresolved fluctuations in streamwise and vertical velocity, and can be termed a 'kinematic momentum flux density'; when multiplied by the reference density ρ_0 it is the unresolved vertical flux density of momentum [$\mathrm{N\,m^{-2}}$]. Let us continue the discussion specifically in the context of the atmospheric boundary layer (with a change in terminology, the same argument works for ocean boundary layers). Since (as observations show) mean windspeed vanishes at ground and gets larger with increasing elevation (implying **mean wind shear**, which is largest near ground), the **covariance $\overline{u'w'}$ normally turns out to be negative** (downward-moving parcels by definition have negative w' and tend to carry excess streamwise momentum relative to the mean velocity at the level considered, such that their $u' > 0$; the contrary argument applies for upward moving parcels).

In the context of a horizontally-uniform flow, we can think about friction by considering layers of the flow. At the top ($z = \delta$) of the hhBL the mean horizontal windspeed attains its free stream value, say $\overline{u}(\delta) = U_\infty$, and by definition the Reynolds stress $\overline{u'w'}(\delta) = 0$. Thus we have a profile $\overline{u'w'} = \overline{u'w'}(z)$ of shear stress across the hhBL, the **shear stress becoming increasingly negative as we approach ground**. The quantity on the r.h.s. of our tendency equation (2), which is the term we call "friction," is (minus) the divergence of the vertical momentum flux carried by unresolved scales, and it is negative. **Any finite layer of our hhBL receives (courtesy of the unresolved motion) a smaller supply of momentum downward across its top face than it loses through its lower face to the flow beneath** (it was a nice touch to give a diagram illustrating this).

As for representation of this friction term in a weather or ocean model, one could base things of a (simplified) budget equation for $\overline{u'w'}$ itself, but this is fairly complex approach. The usual approach in present day models is to invoke the eddy viscosity model, i.e. use the representation

$$\overline{u'w'}(z) = -\,K\,\frac{\partial \overline{u}}{\partial z}$$

which **relates the magnitude and direction of the unresolved vertical momentum flux to the mean wind shear**. A strategy then has to be found to supply the eddy viscosity K [$\mathrm{m^2\,s^{-1}}$]. Commonly K is assumed proportional to the product $\lambda\sqrt{k}$

of a turbulence length scale and the square root of the turbulent kinetic energy k, an approximate budget equation for the latter being included.

2. The analytic solution is $\theta(z) = z(z-1)$. The set-up for numerical solution goes as follows. We label the four values at the gridpoints $\theta_1 ... \theta_4$. The gridlength $\Delta z = 1/3$. We wish to encode into a matrix the four equations:

$$\begin{aligned} \theta_1 &= 1 \\ \theta_1 - 2\theta_2 + \theta_3 &= 2\Delta z^2 \\ \theta_2 - 2\theta_3 + \theta_4 &= 2\Delta z^2 \\ \theta_4 &= 0 \end{aligned}$$

(note the simplicity of the boundary conditions). Thus the matrix formulation of the problem is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/9 \\ 2/9 \\ 0 \end{bmatrix}$$

3. What is called for here is to identify similarities and differences between these two (on the face of it, very different) scientific descriptions of short range dispersion in the neutrally-stratified atmospheric surface layer.

- Similarities between or common points of the two models
  - rather trivially (because it is so obvious), both models are supplied the same parameters that (correctly) characterize the meteorology of the surface layer during PPG57, namely u_*, L and z_0
  - again, trivially, both models incorporate the same (correct) spatial configuration of source and detectors
  - both models neglect the effect of fluctuations u' in streamwise velocity (some students correctly alluded to this approximation as the assumption of "one-way flow")
  - both advect mass alongwind at the correct local mean rate at any height, as encoded by the mean wind profile $U = (u_*/k_v)\ \ln(z/z_0)$
  - both models prohibit any loss of material to ground
  - each of the models is supplied with a flexible or disposable dimensionless constant, the Schmidt number or the Kolmogorov constant, and in each case this is tuned to optimize agreement with PPG57 observations

- Differences between the two models
  - in view of the noted similarities or common points, the essential factor that distinguishes the two models is the treatment of *vertical* motion or vertical spread
  - specifically, the Eulerian model focuses on specifying correctly the mean vertical flux $\overline{w'c'}$ which it models by the eddy diffusion paradigm, viz.

$$\overline{w'c'} = -K(z)\,\frac{\partial C}{\partial z}$$

    whereas the Lagrangian model takes the approach of mimicking a large number of (hopefully) realistic individual (and independent) trajectories. Thus its treatment of the (Lagrangian) vertical velocity W is the key question.

- Why consistency of the two models is not accidental.
  - You were given the hint to consider the effective far field eddy diffusivity implied by G.I. Taylor's analysis (which we had studied in class), namely

$$K^{eff} = \sigma_w^2\,T_L\ . \tag{2}$$

    My intent and hope was that students would compare this K^{eff} of the Lagrangian model, with the eddy diffusivity employed in the Eulerian treatment,

$$K = \frac{k_v}{u_*}\,u_*\,z\ . \tag{3}$$

  - You also had formulae

$$\begin{aligned} T_L &= \frac{2\sigma_w^2}{C_0\,\epsilon} \\ \epsilon &= \frac{u_*^3}{k_v\,z} \\ \sigma_w &= b\,u_* \end{aligned}$$

    that, substituted into Eq. (2) above, yield

$$K^{eff} = \frac{2}{C_0}\,b^4\,k_v\,u_*\,z\ . \tag{4}$$

  - Now we would expect the two models to produce similar results if Eqs. (3, 4) for the eddy diffusivity were equivalent. What would their equality imply? Equating the two diffusivities we get the equation

$$\frac{1}{S_c} = \frac{2}{C_0}\,b^4\ , \tag{5}$$

    which inter-relates the model's tuning parameters C_0 and S_c.
  - If you substitute $C_0 = 3.6$ (i.e. the value we found optimal for the Lagrangian model) into Eqn. (5), you find (since we used $b = 1.3$) that the Eulerian model should match the Lagrangian model if $S_c = 0.63$ — exactly the value we did find optimal