EAS 471 Optional Scored Lab (25%) 2010

Two-layer quasi-geostrophic weather model (Update: March 18, 2010)

Perform a 12 hour weather forecast, deriving initial conditions from a CMC 12 Zulu analysis and comparing forecast solution with the following CMC 00 Zulu analysis. Note that since neither boundary forcing nor surface forcing is admitted, the evolution of any pressure system resolved in the initial state will be highly over-simplified.

Holton (2004, Sec. 8.2) and Haltiner & Williams (1980) both discuss the 'two-layer' baroclinic NWP model based on the quasi-geostrophic vorticity equation and the hydrostatic thermodynamic energy equation. We can regard the dependent variables of the model as being the streamfunction ψ and the vertical velocity ω, because other needed variables can be determined from these. For example the following equations respectively relate the streamfunction to the geopotential (Φ) and the height (h) of a pressure surface, to the vorticity (ζ), and to the Geostrophic velocity components:

$$\begin{aligned}
\Phi &= f_0\,\psi = g\,h \,, \\
\zeta &= \nabla^2\psi \,, \\
U &= -\,\partial\psi/\partial y \,, \\
V &= +\,\partial\psi/\partial x \,.
\end{aligned} \tag{1}$$

Here ∇ is the 2D grad operator and f_0 is the Coriolis parameter evaluated at the central latitude of the domain. Note: in the QG model advection is always computed using the Geostrophic wind, and vertical advection is neglected. The only place the 'true wind' explicitly appears is where its divergence appears in the QG vorticity equation, viz. the term $D_p\, f_0$.

Fig. (1) indicates the structure of the two layer model. Note that use is made of a "staggered grid," that is, different flow variables (more specifically in this case, vertical velocity and vorticity) are represented at different levels: vertical velocity gridpoints are placed on the plane separating the upper and lower layers. Now recall that the continuity equation in pressure coordinates reads

$$D_p + \partial\omega/\partial p = 0 \ , \tag{2}$$

so that in the equations below the terms involving $\omega_2\, f_0/\Delta p$ are exactly this term. Referring to Fig. (1), since ω is non-zero only at level 2, $\partial\omega/\partial p$ is represented by the finite difference $\pm\ (0 - \omega_2)/\Delta p$.

The governing equations, viz. Holton's eqns.(8.5-8.7), can be re-written in flux form by exploiting the fact that the Geostrophic velocity is non-divergent (when defined with fixed $f = f_0$, e.g. Holton p148, 4th edition). The resulting model is:

$$\frac{\partial \zeta_1}{\partial t} = -\frac{\partial}{\partial x}\,(U_1\,\zeta_1\,) - \frac{\partial}{\partial y}\,(V_1\,\zeta_1) - \beta\frac{\partial \psi_1}{\partial x} + \frac{f_0}{\Delta p}\,\omega_2 \tag{3}$$

$$\frac{\partial \zeta_3}{\partial t} = -\frac{\partial}{\partial x}\,(U_3\,\zeta_3\,) - \frac{\partial}{\partial y}\,(V_3\,\zeta_3) - \beta\frac{\partial \psi_3}{\partial x} - \frac{f_0}{\Delta p}\,\omega_2 \tag{4}$$

$$\left(\frac{\sigma\,\Delta p}{f_0}\right)\,\omega_2 = -\frac{\partial}{\partial t}\,(\psi_1 - \psi_3) - \frac{\partial}{\partial x}\,[U_2\,(\psi_1 - \psi_3)] - \frac{\partial}{\partial y}\,[V_2\,(\psi_1 - \psi_3)] \tag{5}$$

where the needed (advecting) Geostrophic velocity at level 2 can be determined by

$$\vec{U}_2 = \hat{k}\,\times\,\nabla\frac{\psi_1 + \psi_3}{2} \tag{6}$$

(linear interpolation of stream function). Note the constant multiplier on the l.h.s. in eqn. (5). The pressure interval $\Delta p \equiv 5 \times 10^4$ Pa, and f_0 is constant. The static stability parameter, in general defined as

$$\sigma = -\alpha\frac{\partial \ln\theta}{\partial p} \tag{7}$$

here takes on a value appropriate to a standard atmosphere under the quasi-geostrophic model. We need σ at the 500 mb level, and Holton quotes $\sigma = 2.5 \times 10^{-6}$ $\mathrm{Pa}^{-2}\,\mathrm{s}^{-2}$ as a mid-tropospheric value. However for the sake of the exercise we shall evaluate it from a sounding in our region of interest, at $t = t_0$.

Notice that eqns. (3, 4) are almost entirely uncoupled — the only connection between levels 1,3 of our model atmosphere is through ω_2, the mid-level vertical velocity. If $\omega_2 > 0$ (sink at the LND) there is a tendency for vorticity aloft at 250 mb to be increasing (for at this level $\partial\omega/\partial p > 0$ meaning $D_p < 0$, negative divergence, ie. shrinking column area) — while the opposite occurs below the LND at 750 mb. Here we may recall Dines' theory of "compensation," viz. the existence of a Level of Non-Divergence (LND) in mid-troposphere (our level 2) separating a lower layer of convergence (divergence) from a compensating upper later of divergence (convergence). From the continuity equation we know that at the LND the vertical velocity takes on its extreme value, thus the logic for placing the gridpoints for ω in the middle of the troposphere.

Now as regards the nature of the numerical solution procedure, it is significant that each of eqns. (3, 4) is of the same *form* as Holton's (p462) forecast problem based on the barotropic vorticity equation, and differs only because we have a term in ω_2 on the r.h.s. Clearly then, if we regard ω_2 as known, we can treat the entire r.h.s. of each of eqns. (3, 4) as a source function, i.e. $F_1(x, y, t)$ and $F_3(x, y, t)$, and carry over the solution method outlined in Holton's Sec. (13.4).

Once we have updated ψ_1, ψ_3, we use the thermodynamic equation (eqn. 5, which explicitly governs the thickness tendency) to diagnose ω_2.

Numerical Method

The instructor's experience in programming this prog is that an explicit method with a forward time step and central spatial differences is unconditionally unstable, and indeed remains unstable even if artificial diffusion is introduced. This is not too surprising, since the linear advection equation is known to be unstable when discretized this way ('Euler discretization'), and our vorticity equations here entail little else than advection.

The problem of instability can be overcome by using a 'leap-frog scheme' or three time level scheme. Thus, the vorticity and stream function are stored at time levels $n-1, n, n+1$ and

$$\left(\frac{\partial\zeta}{\partial t}\right)^n = \frac{\zeta^{n+1} - \zeta^{n-1}}{2\,\Delta t} = F(\text{forcing at time level n}) \tag{8}$$

The forcing term can be evaluated using central differences.

Initialization

We decide on a domain... let it be square, of sidelength $D = 1000$ km. It will be simplest to use equal gridlengths $\Delta x = \Delta y = \Delta$, and let us suppose the origin of the grid is the SW corner ($I = J = 1$). Let $I = 1 ... I_{mx},\ J = 1 ... J_{mx}$ where $I_{mx} = J_{mx} = 10$.

We discretize the height fields within the domain at 750 mb and 250 mb, to get the initial streamfunction, eg.

$$\psi_1(x_I, y_J, 0) = \frac{g}{f_0}\, h_{250}(x_I, y_J) \tag{9}$$

and we also need to get an initial field of ω_2 off the appropriate 500 mb vertical velocity chart. The initial fields of Geostrophic velocity and vorticity are obtained by differentiating the streamfunction (using the usual finite difference approximations).

Boundary conditions

We will need ψ on the boundary. Of course the gradient $\partial\psi/\partial y$ on the western boundary controls the inflow windspeed, thus ψ will be non-zero on the boundary. Perhaps we will treat the boundary value of ψ as being independent of time.

Time-stepping

We had better choose the timestep to satisfy the CFL condition based on $\Delta = 100$ km. Using a simple Euler step, we integrate forward one timestep, eg.

$$\zeta_1(x_I, y_J, t_0 + \Delta t) = \zeta_1(x_I, y_J, t_0) + \left(\frac{\partial \zeta_1}{\partial t}\right)_{t_0} \Delta t \tag{10}$$

Now, knowing the new vorticity $\zeta_1(x_I, y_J, t_0 + \Delta t)$ we need to solve

$$\nabla^2 \psi_1 = \zeta_1(x, y, t_0 + \Delta t) \tag{11}$$

to get the updated streamfunction... knowing which allows us to differentiate to get the velocity. Eqn. (11) is diagnostic, and constitutes an elliptic (ie. "jury") problem. ζ is a source function. There is a straightforward iterative method to get $\psi_1(x_I, y_J, t_0 + \Delta t)$, viz. the "relaxation method."

References

Haltiner, G.J., & Williams, R.T. 1980. *Numerical Prediction and Dynamic Meteorology*. J. Wiley.

Holton, J.R. 2004. *An Introduction to Dynamic Meteorology (Fourth Edition)*. Elsevier.

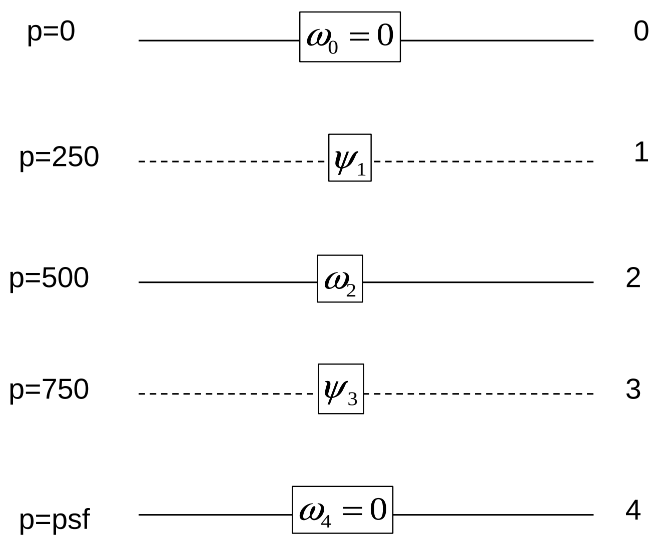

Fig.(8.2) of Holton (2004). Organization of the dependent variables in two-level baroclinic model

Figure 1: On each plane, visualize a discrete (gridded) field of the associated variable, eg. $\psi_1(x_I, y_J)$.