EAS 471, Optional 2^{nd} scored Cmpt'g Assignment, 2010

Spectral solution of the non-linear 1D advection equation

The *non*-linear advection equation, also known as the inviscid (frictionless) one-dimensional Burgers' equation[1] is

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = 0 . \tag{1}$$

Adopting the spectral method, solve this equation on $-1 \le x \le 1$, with periodic boundary condition $U(-1,t) = U(1,t)$ and initial condition

$$U(x,0) = U_0(x) = \cos(\pi x) . \tag{2}$$

Theory underlying the spectral method

The trial solution is

$$u(x,t) = \sum_{-N}^{N} u_n(t)\, e^{j\, n\pi x} \tag{3}$$

where $u_n(t)$ is the (complex) amplitude of the n^{th} wave, and note that the u_n carry the time-dependence of $u(x,t)$. Substituting into the advection equation, the error function is:

$$e(x) = \sum_{n=-N}^{N} \frac{du_n}{dt} e^{j\, n\pi x} + \sum_{n=-N}^{N} u_n(t)\, e^{j\, n\pi x} \sum_{m=-N}^{N} jm\pi\, u_m(t)\, e^{j\, m\pi x} \tag{4}$$

[1] The one-dimensional Burgers' equation

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = \epsilon \frac{\partial^2 U}{\partial x^2}$$

is a time-dependent advection-diffusion equation. Here ϵ is the viscosity, and in the inviscid limit $\epsilon \to 0$.

To determine the optimal coefficients $u_n(t)$, multiply the error through by $e^{-j\,p\,\pi\,x}$ and integrate over the range of x, requiring

$$\int_{-1}^{1} e(x)\; e^{-j\,p\,\pi\,x}\; dx = \; 0 \qquad\qquad p = -N, ...N \tag{5}$$

Then

$$\begin{aligned} 0 \;&=\; \sum_{n=-N}^{N} \frac{du_n}{dt} \int_{-1}^{1} e^{-jp\pi x}\; e^{jn\pi x}\; dx \\ &+\; \sum_{n=-N}^{N} u_n(t) \sum_{m=-N}^{N} \frac{jm\pi}{L}\, u_m(t) \int_{-1}^{1} e^{jn\pi x}\; e^{jm\pi x}\; e^{-jp\pi x}\; dx \end{aligned} \tag{6}$$

But we have an orthogonality rule (Spiegel, Advanced Mathematics, p187)

$$\begin{aligned} &\int_{-1}^{1} e^{jm\pi x}\; e^{-jp\pi x}\; dx \\ =\;& \int_{-1}^{1} (\; \cos m\pi x + \; j\; \sin m\pi x\;)\; (\; \cos p\pi x - \; j\; \sin p\pi x\;)\; dx \\ =\;& 2\; \delta_{m,p}\; . \end{aligned} \tag{7}$$

Exploiting this rule it can be shown that the rate of change of the coefficients is

$$\frac{du_p}{dt} = \; - \sum_{n=-N}^{N} \sum_{m=-N}^{N} jm\pi\; u_n(t)\; u_m(t)\; \delta_{m+n,p}\; . \tag{8}$$

This is a set of $2N+1$ coupled ordinary differential equations for the advancement in time of the complex coefficients $u_p(t),\; p = -N.....N$.

Splitting the coefficients into their real and imaginary parts,

$$u_n(t) = u_n^R(t) + j u_n^I(t) \tag{9}$$

it is straightforward to show that

$$\frac{du_p^R}{dt} = \sum_{n=-N}^{N} \sum_{m=-N}^{N} m\pi \left[u_n^I(t)\, u_m^R(t) \;+ u_m^I(t)\, u_n^R(t)\,\right] \delta_{m+n,p}$$

$$\frac{du_p^I}{dt} = \sum_{n=-N}^{N} \sum_{m=-N}^{N} m\pi \left[u_n^I(t)\, u_m^I(t) - \; u_n^R(t)\, u_m^R(t)\,\right] \delta_{m+n,p}\,. \quad (10)$$

Numerical Method

The domain size (a length of 2) and the maximum initial velocity (1) imply a timescale. The timestep Δt can be chosen arbitrarily — the Courant condition does not apply since we do not discretize in x —- but should be small with respect the timescale. Pick an arbitrary number of waves, e.g. to start off with, $N = 5$.

Eqns. (10) are used to advance the coefficients, e.g.

$$u_p^R(t + dt) = u_p^R(t) + \frac{du_p^R}{dt}\, \Delta t\,, \quad (11)$$

and at any time the solution is constructed as

$$u(x,t) = \sum_{-N}^{N} u_n^R(t) \cos n\pi x - \; u_n^I(t) \sin n\pi x\,. \quad (12)$$

The initial velocity field must be represented by the decomposition:

$$u(x,0) = \sum_{n=-N}^{N} u_n(0)\, e^{jn\pi x}\,. \quad (13)$$

In our chosen case we initialize with a single long cosine wave corresponding to $n = 1$, ie. $u(x,0) = \cos(\pi x)$. Thus only the coefficient u_1 is non-zero at commencement of the integration (ie. $u_1(0) \neq 0$), with $u_1^R(0) = 1$ and $u_1^I(0) = 0$.