Professor: J.D. Wilson   Time available: 60 mins   Value: 15%

Multi-choice $(12 \times \frac{1}{2}\% = 6\%)$

1. Suppose ϕ represents the quantity per unit volume of a certain scalar property of the air, and $\mathbf{u}$ the velocity field. Which choice correctly gives the convective flux density $\mathbf{F}$ of ϕ?
   (a) $\mathbf{F} = \mathbf{u} \cdot \nabla\phi$
   (b) $\mathbf{F} = \mathbf{u}\,\phi$ ✓✓
   (c) $\mathbf{F} = \mathbf{u}\,\nabla\phi$
   (d) $\mathbf{F} = \phi\,\nabla \cdot \mathbf{u}$
   (e) $\mathbf{F} = \mathbf{u} \times \nabla\phi$

2. Suppose vectors $\mathbf{u}$ and $\mathbf{v}$ have representation $\mathbf{u} = (\alpha, 0, 1)$ and $\mathbf{v} = (0, 1, \beta)$ when projected onto orthogonal unit vectors $(\hat{i}, \hat{j}, \hat{k})$. Which option gives their cross product $\mathbf{u} \times \mathbf{v}$?
   (a) $\alpha\hat{i} - \beta\hat{k}$
   (b) 1
   (c) $-\hat{i} - \alpha\beta\hat{j} + \alpha\hat{k}$ ✓✓
   (d) $\beta\hat{k}$
   (e) $\alpha\beta$

3. What is the wavelength of the shortest wave that can be represented on a grid whose interval is Δx?
   (a) $\pi/\Delta x$
   (b) $2\pi/\Delta x$
   (c) $\Delta x/2$
   (d) Δx
   (e) $2\Delta x$ ✓✓

4. Suppose that an odd function $\theta(x)$ defined on the range $-1 \le x \le 1$ is represented by a Fourier series

$$\theta(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where (as can be shown) the coefficients are defined by integrals

$$b_m = \int_{-1}^{1} \theta(x) \sin(m\pi x) \; dx \,. \tag{1}$$

Which interpretation is **false**?

(a) b_m is the average value of $\theta(x)$ along the interval $-1 \le x \le 1$ ✓✓

(b) b_m is essentially the correlation of the signal $\theta(x)$ with $\sin(m\pi x)$

(c) b_m is twice the average value along the interval $-1 \le x \le 1$ of the product of the signal $\theta(x)$ and the "mode" (wave) $\sin(m\pi x)$

5. The concentration $\phi(\mathbf{x}, t)$ of a certain property is governed by

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{F} + Q(\mathbf{x}, t) \,.$$

Which term in the equation is a "transport term"?

(a) $\partial\phi/\partial t$

(b) Q

(c) $\mathbf{F}$

(d) $-\nabla \cdot \mathbf{F}$ ✓✓

6. Which is the correct expression of the quantity $\nabla \cdot \nabla\phi$ in Cartesian coordinates?

(a) zero

(b) $\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial z}$

(c) $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}$

(d) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ ✓✓

(e) 1

7. Suppose $\phi(x,t)$ is defined on $-\infty \leq x \leq \infty$, and is governed by the 1D linear advection equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = 0$$

where $U =$ const. If $\phi(x,0) \equiv \phi_0(x) = \sin(kx)$ where k is the wavenumber, which option gives $\phi(x,T)$?

(a) $\sin[k(x - UT)]$ ✓✓

(b) $\sin(kx) - UT$

(c) $\phi_0(x) + \sin(kx)$

(d) $\phi_0(x) + UT$

8. Suppose the 1D heat equation were discretized as

$$\frac{T_I^{n+1} - T_I^n}{\Delta t} = \kappa \, \frac{T_{I+1}^n + T_{I-1}^n - 2T_I^n}{\Delta x^2} \, , \tag{2}$$

where I labels gridpoints on the x-axis separated by distance Δx, etc. Which option correctly states the order of the truncation error?

(a) $\mathrm{O}[\Delta t]$

(b) $\mathrm{O}[\Delta x^2]$

(c) $\mathrm{O}[\Delta t^2] + \mathrm{O}[\Delta x^2]$

(d) $\mathrm{O}[\Delta t^2] + \mathrm{O}[\Delta x]$

(e) $\mathrm{O}[\Delta t] + \mathrm{O}[\Delta x^2]$ ✓✓

9. Suppose (again) the 1D heat equation were discretized by Eq. (2). What name is given to the quantity $\kappa \Delta t / \Delta x^2$, and what are its units?

(a) Courant number; dimensionless

(b) diffusion number; dimensionless ✓✓

(c) thermal diffusivity; $\mathrm{m^2\,s^{-1}}$

(d) tridiagonal matrix coefficient; dimensionless

(e) kinematic heat flux; $\mathrm{K\,m\,s^{-1}}$

10. Suppose particles are released independently from $z = 0$ at time $t = 0$. If their mean-square displacement on the z axis at arbitrary time t is

$$\langle z^2 \rangle = 2\,K\,t$$

for all t, then certain inferences can be drawn. Which is **false**?

(a) the particles' motion is a superposition of convection and diffusion ✓✓

(b) a statistically correct ensemble of particle trajectories $Z = Z(n\Delta t)$ could be computed using the drunkard's walk (Δt the time step)

(c) the random distance steps of the drunkard's walk should obey $dZ = \sqrt{2K\,dt}\;r$ where r has unit variance and vanishing mean

(d) K is the effective diffusivity of the transport process

11. Let $p(z, t|z_0, t_0)$ be the transition density function (or particle displacement probability density function). Which option gives the probability that at time t a particle that had been released at $z = z_0$ at earlier time t_0 will lie within an elementary layer $z_1 \pm dz/2$?

(a) $p(z_1, t|z_0, t_0)$

(b) $p(z_1, t|z_0, t_0)\;dz$ ✓✓

(c) $p(z_1, t|z_0, t_0)\;dz/2$

(d) $p(z_1, t|z_0, t_0)\;(t - t_0)$

(e) $p(z_1, t|z_0, t_0)\;(t - t_0)\;dz$

12. In "advection form" the continuity equation reads

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho\,\nabla \cdot \mathbf{u}$$

Which of the following alternative ways of writing the continuity equation is called its "transport form"? (D/Dt denotes the Lagrangian derivative.)

(a) $\frac{D\rho}{Dt} = -\rho\,\nabla \cdot \mathbf{u}$

(b) $\frac{D\ln\rho}{Dt} = -\,\nabla \cdot \mathbf{u}$

(c) $\frac{\partial \rho}{\partial t} = -\,\nabla \cdot \rho\mathbf{u}$ ✓✓

(d) $\frac{1}{\rho}\,\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0$

Short answer: 2 x 4.5% = 9%

Answer any **two** questions from this section. [Schematic answers given below].

1. Suppose an individual had been given a finite difference code purporting to model the time-average, crosswind-integrated concentration field $\chi(x, z)$ due to a continuous (i.e. steady) point source of passive gas in the ASL. With what seems an adequate choice of gridlengths $\Delta x, \Delta z$ the code is producing plausible-looking output, but the individual lacks any data to verify these results. Describe how s/he might provide an assurance — i.e. legitimately claim to those questioning the matter — that (in a certain limit) the numeric solution represents the true (albeit unknown) field of χ. (Hint: the Lax Equivalence Theorem is relevant.)

2. The task for lab assignment 1 was to solve the advection-diffusion equation

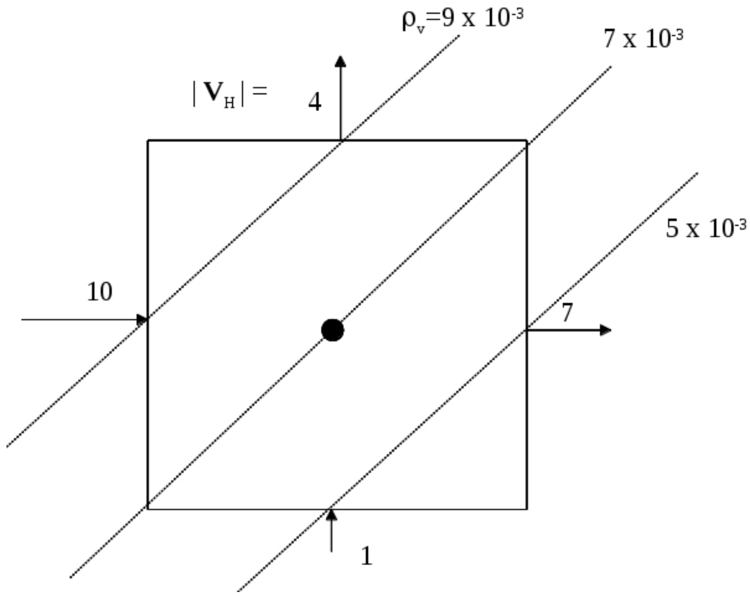
$$U(z)\,\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left[K(z)\,\frac{\partial C}{\partial z}\right] + Q$$

for the concentration field $C(x, z)$ due to a continuous line source of gas near ground in the atmospheric surface layer, with profiles

$$\begin{aligned} U &= \frac{u_*}{k_v}\,\ln\frac{z}{z_0}\,, \\ K &= (k_v/S_c)\,u_* z\,. \end{aligned}$$

Draw a diagram that illustrates the geometry of the problem (i.e. axes, location and orientation of source, etc). Describe specifically the assumptions or simplifications or restrictions underlying this model, classifying them as to whether they fundamentally address the *micrometeorology* of the ASL (i.e. wind and temperature fields) or whether their scope is more narrowly the field of C. Define any symbols you introduce.

3. The diagram below shows a square cell (a 2D "control volume") with side-length 10 km. Arrows with affixed numbers give the direction and magnitude [$\mathrm{m\,s^{-1}}$] of the horizontal wind vector $\mathbf{V}_H$ at each interface, and the diagonals are contours of absolute humidity ρ_v [$\mathrm{kg\,m^{-3}}$]. For this cell calculate: the Laplacian of the humidity, $\nabla^2\rho_v$; the velocity divergence, $\nabla_H \cdot \mathbf{V}_H$; and the rate of moisture accession $\nabla_H \cdot (\rho_v\,\mathbf{V}_H)$.



Definitions relating to the atmospheric surface layer

Monin-Obukhov similarity theory (MOST) applies to a horizontally-homogeneous atmospheric surface layer.

- friction velocity u_*, often defined

$$u_*^4 = \sqrt{\overline{u'w'}^2 + \overline{v'w'}^2} \ ,$$

  gives mean drag $\tau = \rho u_*^2$ [N m^{-2}] $= \sqrt{\tau_x^2 + \tau_y^2}$ of the wind on the ground

- dimensionless gradient in mean wind speed

$$\frac{k_v z}{u_*} \frac{\partial \overline{u}}{\partial z} \equiv \frac{k_v}{u_*} \frac{\partial \overline{u}}{\partial \ln(z)} = \phi_m \left(\frac{z}{L} \right) \ .$$

  The universal Monin-Obukhov function on the r.h.s. has the limit $\phi_m(0) = 1$ in neutral stratification, ie. at $|z/L| = 0$. Similar expressions apply for the gradients in mean potential temperature, humidity, carbon dioxide concentration, etc. The von Karman constant $k_v = 0.4$, u_* is the friction velocity and L is the Obukhov length

- Obukhov length, defined

$$L = - \frac{u_*^3 \, T_0}{k_v \, g \, \overline{w'T'}}$$

  where T_0 [K] is the mean temperature of the layer and $\overline{w'T'}$ is the (kinematic sensible) heat flux (density)

Elements for responses to short-answer questions

1. One would have to ascertain that the discretized equations solved by the code correspond to an acceptable scientific model of the process of gas dispersion in the ASL, such as (for instance) an advection-diffusion equation with suitable profiles of advecting wind speed and of the eddy diffusivity. One would admit to the skeptic that the advection-diffusion equation is inexact, in the sense that assumptions and simplifications are involved in its derivation from the mass conservation equation (see question 2); but one would make the appeal that modelling *almost always* entails some degree of approximation, and argue (if appropriate) that this underlying paradigm is an adequate model. This element was not addressed by any of the students responding, but it is by no means a trivial issue.

   Once one had ascertained that the differential equation(s) underlying the code *are* acceptable, such that their (unknown) solution $\chi(x, z)$ can be considered the closest thing available to the "truth," there remains the question of whether *numeric* solutions $\widetilde{\chi}(x, z)$ provided by the procedure (code) adequately approximate that truth. This is where the Lax Equivalence theorem resolves debate: for it assures us that provided (a) that our algorithm is numerically stable and (b) that the discretized equation(s) is/are *consistent* with the underlying differential equation(s), then in the limit that the gridlengths $\Delta x, \Delta z$ approach zero the numeric solution approaches the unknown true solutoin, i.e. $\widetilde{\chi} \to \chi$. (One could also explain what "consistency" means in this context: truncation error vanishes as the gridpenths go to zero.) All students who responded laid this out quite well – though not everyone stated the Lax Equivalence Theorm precisely.

2. Some responses wandered far away from what was actually asked for: begin by *comprehending* the question. It *does not* ask for a definition of the ASL, nor about its character: it addresses the *model*. It does not ask that the solution C be displayed. It does not ask for definitions (other than those of any new symbols introduced in responding). It does not ask for the specific values of u_*, z_0, S_c that were used in the simulations, nor for the gridlengths.

   Assumptions implicit in the model that address the micrometeorology, and underpin the adopted (Monin-Obukhov) profiles of mean wind speed and eddy diffusivity:

   1) stationarity and
   2) horizontal-homogeneity of the velocity statistics
   3) adiabatic (unstratified) flow
   4) unidirectional mean flow, with the x-axis chosen so as to be aligned with the mean wind direction

   Simplications potentially affecting the quality of the solution for C:

   5) neglect of the alongwind eddy flux of mass $\overline{u'c'}$ (or rather, of its along-wind gradient $\partial\overline{u'c'}/\partial x$
   6) adoption of a gradient-diffusion model

   $$\overline{w'c'} = -K\,\frac{\partial C}{\partial z}$$

      for the vertical eddy flux of mass.

   (Items 1,2,3,5,6 mandatory for full marks on this question.)

3. Computations result in:

   - Laplacian of humidity $\nabla^2\rho_v = 0$ [$\mathrm{kg\,m^{-5}}$]
   - Velocity divergence $\nabla_H \cdot \mathbf{V}_H = 0$ [$\mathrm{s^{-1}}$]
   - Rate of moisture accession $\nabla_H \cdot (\rho_v\,\mathbf{V}_H) = -2.4 \times 10^{-6}$ [$\mathrm{kg\,m^{-3}\,s^{-1}}$].