EAS 471 $2^{\rm nd}$ Scored Assignment (20%) Due: 17 Mar. 2016

Conduction in the plane

The heat equation for a steady state problem in two dimensions may be written

$$\frac{\partial T}{\partial t} = 0 = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T + Q \,, \tag{1}$$

where κ is the molecular thermal diffusivity (assumed constant) and Q is the volumetric source distribution. We shall suppose there is a steady heat source localized at the origin[1], such that Eq. (1), an instance of "Poisson's equation", becomes

$$0 = \kappa \, \nabla^2 T + \alpha \, \delta(x - 0) \, \delta(y - 0) \,. \tag{2}$$

This gives the steady-state temperature field $T(x, y)$ resulting from the source (whose strength is α, $\mathrm{K\,m^{-2}}$). Let the domain be the unit square $-1/2 \le x, y \le 1/2$ and let the boundary temperature be $T = 0^{\rm o}$C. For simplicity let the thermal diffusivity $\kappa = 1$ and let the heat source strength $\alpha = 1$.

Solve this Poisson's equation using an iterative "relaxation method" and the standard computational molecule for the 2-D Laplacian. Set up your grid with uniform mesh $\Delta x = \Delta y = \Delta$ such that gridpoint I = J = 0 falls at the origin: only at this gridpoint is the source term (numerical delta function) non-zero, and it must be set equal to $1/\Delta^2$. Study the impact of two choices for the resolution, viz. $\Delta = (0.05, 0.25)$. Compare your numerical solutions with the analytical solution[2],

$$T(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{4 \, \pi^{-2}}{(2m+1)^2 + (2n+1)^2} \, \cos\left[(2m+1)\,\pi x\right] \, \cos\left[(2n+1)\,\pi y\right] \tag{3}$$

(carrying the summation to $m, n = 100$ will more than suffice, since these limits correspond to very high wave number contributions to the solution).

Iterate your solution until the largest residual $\max(|R_{i,j}|) \le 10^{-7}$. Remember you may plot your analytical solution with arbitrarily high spatial resolution, ie. as a line, for it is a spatially-continuous solution. However the numerical solution is known (and should therefore be plotted) *only at the gridpoints.*

[1] Note that the δ-function has units: for present purposes we may define the delta-function according to: $\int_{-\infty}^{\infty} f(x) \, \delta(x - x_1) \, dx = f(x_1)$

[2] The analytical solution has been obtained using a "boundary-weighted Galerkin method". Cosine functions have been chosen as basis functions, for they satisfy the required symmetry about the origin, and vanish at the boundary of the unit square; and in view of the use of cosine functions, one can equivalently view the solution as stemming from the Fourier series approach. In terms of connection with methods in PDEs, the analytical solution can be said to be the Green's function for the Laplacian in the unit square (for the particular size and boundary condition assumed).