EAS 471, Atmospheric Modelling

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Contents

Chapter 1

General Character of Flow Equations

1.1 Eulerian Derivation of a Generalized Conservation Equation

Rather than commencing our study with, say, the Navier-Stokes equations which you may have encountered in another course, we will begin by deriving a generalized conservation equation for two-dimensional flow — the generalization to three dimensions is mechanical.

Let $\phi = \phi(x, z, t)$ denote the amount of a certain property per unit volume of the fluid, eg. if we have a 'mass' property then ϕ has the units [kg m^{-3}]. Our independent variables are (x, z, t) and ϕ is a "field." We derive a conservation equation for ϕ by performing a ϕ-budget on a "control volume" (cv), an imaginary surface (see Fig. 1.1) through which the fluid freely moves, having dimensions $\Delta x, \Delta z$. Both diffusion and convection can transfer "ϕ" across the control volume (cv) walls. Through all space, the property is (or may be) created at a rate Q [kg m^{-3} s^{-1}] that is entirely arbitrary.

At any time the amount in the cv is $\phi(t)\Delta x\Delta z$ (where ϕ and Q are to be considered average values throughout the control volume). This changes over time Δt by an amount $\Delta\phi\ \Delta x\ \Delta z$ due to internal production and/or net exchange across the control volume faces.

Let F_x, F_z be the components of the *flux density* of ϕ along x and along z, [kg m^{-2} s^{-1}]. Then logic demands[1] that

$$\begin{aligned} \Delta\phi\Delta x\Delta z &= \Delta z\Delta t\,[F_x(x) - F_x(x+\Delta x)] \\ &+ \Delta x\Delta t\,[F_z(z) - F_z(z+\Delta z)] + Q\Delta x\Delta z\Delta t \end{aligned} \tag{1.1}$$

where the fluxes are considered average values along the (small) faces. Now divide by $\Delta x\Delta z\Delta t$ and let $(\Delta x, \Delta z, \Delta t)$ all become infinitesimal. We obtain the partial differential equation

$$\frac{\partial\phi}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_z}{\partial z} + Q\ , \tag{1.2}$$

[1] If you ponder the matter, I think you'll see that this equation *has* to be true by definition of the symbols it uses. It does not require proof.

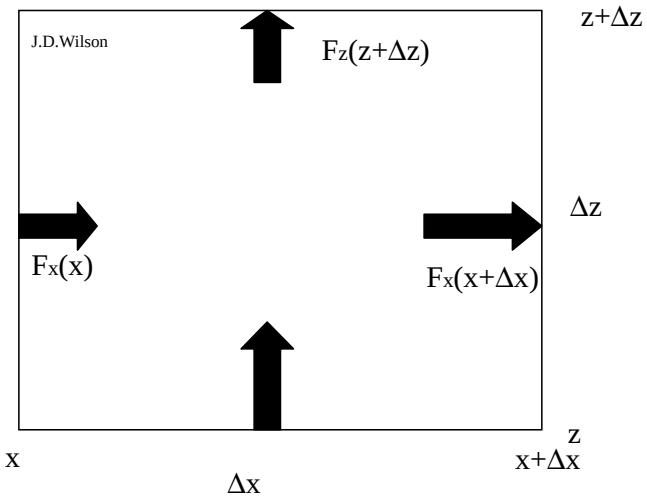

Figure 1.1: *An imaginary control volume through which the fluid flows freely, and the interfacial fluxes of "ϕ" carried by the flow. Arrows indicate the direction of positive flux.*

or generalising to 3d:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} + Q \ . \tag{1.3}$$

Eq. (1.3) may be written in more compact form using tensor notation

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F_i}{\partial x_i} + Q \ , \tag{1.4}$$

or vector notation as in

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \vec{F} + Q \ . \tag{1.5}$$

In Eq. (1.5) ∇ is the "grad operator," whose expression in Cartesian coordinates is

$$\nabla \equiv \ \hat{i}\ \frac{\partial}{\partial x} + \ \hat{j}\ \frac{\partial}{\partial y} + \ \hat{k}\ \frac{\partial}{\partial z} \ . \tag{1.6}$$

The first term on the right hand side (r.h.s.) of Eq. (1.5) is the "divergence" of the flux vector. The local rate of change in time (local tendency) of the concentration of a fluid attribute is driven by the divergence of the flux of that attribute, summed with internal (i.e. volumetric, or distributed) production[2].

[2]An example of a volumetric production term is the exchange of water between liquid and vapour phase.

1.2 Explicit form for the flux

Depending on what specifically ϕ represents, the physical mechanisms that may transport this quantity (i.e. contribute to the flux of ϕ) are: radiation, convection and diffusion. Assuming ϕ is a material concentration we need consider only the latter two transport mechanisms[3], and we may write the flux vector more explicitly as

$$\vec{F} = \vec{u}\,\phi - \ K\,\nabla\phi\ , \tag{1.7}$$

where K is the diffusivity [$\mathrm{m}^2\ \mathrm{s}^{-1}$] (of the property labelled ϕ in the fluid medium). Substituting, we find

$$\frac{\partial\phi}{\partial t} = -\nabla\cdot(\,\vec{u}\,\phi - \ K\,\nabla\phi\,) + Q \tag{1.8}$$

which (if we treat the diffusivity as independent of position, and assume the velocity field to be non-divergent[4]) rearranges to

$$\frac{\partial\phi}{\partial t} + \ \vec{u}\cdot\nabla\phi = K\nabla^2\phi + \ Q\ . \tag{1.9}$$

This is a time-dependent advection-diffusion equation; and as you may anticipate, we can simplify the l.h.s. by adopting the Lagrangian time derivative (Sec. 1.3).

1.2.1 Review of the grad operator

∇ operates on a scalar "directly," eg. ∇T is the vector gradient in temperature. The grad operator operates on a vector "indirectly," by means of an "inner product" or "dot product", eg. the velocity divergence

$$\begin{aligned}\nabla\cdot\vec{u} \ &\equiv \ \left(\hat{i}\,\frac{\partial}{\partial x} + \ \hat{j}\,\frac{\partial}{\partial y} + \ \hat{k}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\,u + \ \hat{j}\,v + \ \hat{k}\,w\right)\\ &= \ \frac{\partial u}{\partial x} + \ \frac{\partial v}{\partial y} + \ \frac{\partial w}{\partial z}\end{aligned} \tag{1.10}$$

(an alternative notation for the velocity divergence is $\mathrm{div}\,\vec{u}$, so $\mathrm{div}\equiv\nabla\cdot$).

1.3 The Lagrangian (Material) Derivative

Let $q = q(x,t)$ be any continuum property of the fluid. Then by the chain rule of differentiation, the differential

$$dq = \left(\frac{\partial q}{\partial t}\right)_x\ dt + \left(\frac{\partial q}{\partial x}\right)_t\ dx\ . \tag{1.11}$$

[3]Holton (2004) derives specifically the conservation equation for air mass (the "continuity equation"), ie. $\phi\to\rho$. "Air" is neither created nor destroyed in situ, so there is no Q term; and because air does not diffuse in air, there is no diffusive flux and so the flux of air is purely convective: $\vec{F} = \vec{u}\,\rho$.

[4]i.e. that $\nabla\cdot\vec{u} = 0$.

Now constrain dx, dt such that $dx/dt = u$, the velocity. Dividing by dt we have

$$\frac{dq}{dt} = \frac{\partial q}{\partial t}\, dt + \, u\, \frac{\partial q}{\partial x}\,, \tag{1.12}$$

or, generalizing to three space dimensions,

$$\frac{dq}{dt} = \frac{\partial q}{\partial t}\, dt + \, u\, \frac{\partial q}{\partial x}\,, + v\, \frac{\partial q}{\partial y} \, + \, w\, \frac{\partial q}{\partial z}\,. \tag{1.13}$$

This is the rate of change of q *moving with the flow*, or in more technical terms, *following a fluid element* (you can see that this is so, because we constrained the movements dx, dy, dz such that the vector displacement $\mathbf{dx} = \mathbf{u}\, dt$).

Now having introduced the Lagrangian derivative we have the option to express our flow equations in a simpler form. In particular the advection-diffusion equation (Eq. 1.9) simplifies to

$$\frac{d\phi}{dt} = K\nabla^2\phi + \, Q\,. \tag{1.14}$$

Interestingly this says that if we could neglect molecular transport, and if there were no internal sources/sinks, then the property ϕ would be conserved along fluid element paths.

1.4 Classification of terms

The terms appearing in conservation equations (such as Eqs. 1.3, 1.8, 1.9) can be classified as:

- *Storage Terms*: The local time rate of change of the "content" of ϕ, $\partial\phi/\partial t$.
- *Transport Terms*: Terms of form $\partial()/\partial s$ where s is any space coordinate. Such a term is a "transport term" because, upon integration with respect to a coordinate (eg. x) from X_1 to X_2, it reduces to a difference between the influx at X_1 and efflux at X_2. Such terms cause redistribution (of ϕ, or whatever) within the flow domain, but no destruction/creation. Sometimes a term which does not appear to be a transport term can, by manipulation of the equation, be transformed to an explicit transport term (this is true, for example, of the advection term). Particularly in the context of numerical modelling, it is advantageous to cast advection terms in flux form. This transformation involves the continuity equation. When flow equations are cast in "flux form" it is easier to derive discretisation procedures which conserve properties which should (by inspection of the corresponding differential equations and b/conds) be conserved.
- *Source/sink Terms*: All terms not of the above two categories are production or destruction terms

Note that the presence of a source term Q in general renders a conservation equation inhomogeneous[5]. Typically in fluids problems the dependent variable is "forced" either by an inhomogeneity (source) in its conservation equation, or by the nature of the boundary- or initial-conditions (eg. specification of an inward flux across an upwind boundary on an axis having the '1-way' property defined later).

1.5 Discretization of the Conservation Equations

1.5.1 Simplest "Finite Difference" Methodology

Real flow problems involve variables which are spatially continuous (and have spatially continuous derivatives). But in seeking a numerical solution to a flow problem, we cannot retain continuous information on the distribution of the dependent variable: knowledge of the values ϕ at discrete points in space and time must suffice. (Aside: if I choose to represent a variable as a wave such as a sine wave, I can know its value at all points; this is the "spectral" approach to discretization, and in this case one adopts a finite set of waves, which when superimposed, represent approximately, the desired spatial variation. More on this later.)

But then how do we approximate a differential equation (which after all expresses our fundamental knowledge, our conservation principle)? A derivative $\partial\phi/\partial x$ is the limit (as $\Delta x \to 0$) of the ratio $\Delta\phi/\Delta x$. We cannot shrink $\Delta x \to 0$ if we only know ϕ at discrete points along the x axis.

Patankar (1980, pp26-31) gives an excellent discussion of "discretization," and Ames (1977, pp 15-19) more thoroughly discusses the particular discretization method covered here. Note that however we accomplish discretization, unless we adopt a spectral method (wherein spatial variation is represented by a superposition of waves) the outcome will be a set of "difference equations" or "neighbour equations," which relate the value ϕ_C of our variable at gridpoint C and time t to its neighbours in space and time.

What are usually called "Finite-Difference" methods for obtaining the discretized equations (ie. the neighbour equations) are obtained by approximating derivatives with a truncated Taylor series

$$\begin{aligned}
\phi(x+\Delta x) &= \phi(x) + \left(\frac{\partial\phi}{\partial x}\right)_x \Delta x + \frac{1}{2!}\left(\frac{\partial^2\phi}{\partial x^2}\right)_x \Delta x^2 + \frac{1}{3!}\left(\frac{\partial^3\phi}{\partial x^3}\right)_x \Delta x^3 + ... \\
\phi(x-\Delta x) &= \phi(x) - \left(\frac{\partial\phi}{\partial x}\right)_x \Delta x + \frac{1}{2!}\left(\frac{\partial^2\phi}{\partial x^2}\right)_x \Delta x^2 - \frac{1}{3!}\left(\frac{\partial^3\phi}{\partial x^3}\right)_x \Delta x^3 + ...
\end{aligned} \tag{1.15}$$

[5] A linear ordinary differential equation has the form $a_0(x)\,\frac{d^n y}{dx^n} + a_1(x)\,\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\,\frac{dy}{dx} + a_n(x)\,y = Q(x)$, and it is a homogeneous o.d.e. if $Q(x) = 0$.

For example we can approximate $\partial\phi/\partial x$ at x as:

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\phi(x+\Delta x)-\phi(x)}{\Delta x}+O[\Delta x]\,,\\ \frac{\partial\phi}{\partial x} &= \frac{\phi(x)-\phi(x-\Delta x)}{\Delta x}+O\left[\Delta x\right]\,,\\ \frac{\partial\phi}{\partial x} &= \frac{\phi(x+\Delta x)-\phi(x-\Delta x)}{2\Delta x}+O\left[\Delta x^2\right]\end{aligned} \tag{1.16}$$

which are called respectively the *forward, backward*, and *central* differences; these approximations are sometimes termed "computational molecules" for the first derivative. The central difference is appealing because its error of approximation is of higher order thus (for small Δx) is smaller.

Similarly, the curvature $\partial^2\phi/\partial x^2$ at x may be approximated:

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\phi(x+\Delta x)+\phi(x-\Delta x)-2\,\phi(x)}{\Delta x^2}+O\left[\Delta x^2\right]\,. \tag{1.17}$$

Ames (1977, p17) gives a pictorial view of these (and more complex) computational molecules.

1.5.2 The Laplacian Operator

In a Cartesian coordinate system the "Laplacian operator" (or "curvature" or "diffusion" operator) has the representation

$$\nabla^2\equiv\frac{\partial^2}{\partial x_j\,\partial x_j} \tag{1.18}$$

where the Einstein summation convention is implied; expanding the summation,

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \tag{1.19}$$

and (e.g.) in 1-dimension, $\nabla^2=\partial^2/\partial x^2$.

The Laplacian is a smoothing operator. First, let us note from eqn (1.17) that its simplest computational molecule can be said to compare the "central" value with its neighbours. If the central value is "too large" relative to the average of its neighbours, the term will be negative, and vice versa.

Consider the action of the Laplacian in the 1-d heat equation[6]

$$\frac{\partial T}{\partial t}=K\,\frac{\partial^2 T}{\partial x^2} \tag{1.20}$$

where the right-hand side can be seen as the divergence (in 1-d) of a conductive heat flux in a system with constant molecular thermal diffusivity K:

$$\frac{\partial T}{\partial t}=-\frac{\partial}{\partial x}\left(-K\,\frac{\partial T}{\partial x}\right)\,. \tag{1.21}$$

[6]We could equally well call this a 1-d "diffusion equation".

Suppose we represent the temperature at discrete points (labelled I) along the x-axis, these points being separated by a constant gridlength Δx. Our other independent variable is t, which we also discretize, on intervals Δt such that $t^n = n\ \Delta t$. Then ${T_I}^n$ is the temperature at the I^{th} gridpoint at time n.

A possible discretisation of the equation is:

$$\frac{T_I^{n+1} - T_I^n}{\Delta t} = K\ \frac{T_{I+1}^n + T_{I-1}^n - 2\ T_I^n}{\Delta x^2} \tag{1.22}$$

which on rearrangement gives the convenient formula:

$$T_I^{n+1} = T_I^n + \frac{K\ \Delta t}{\Delta x^2}\ \left(T_{I+1}^n + T_{I-1}^n - 2\ T_I^n\right)\ . \tag{1.23}$$

This is called an "explicit" algorithm, because its form is "single unknown = combination of knowns". It is now clear that the diffusion (∇^2) term is a smoother; the new value of T at a given point is bigger (smaller) than the preceding value depending on whether that preceding value is smaller (bigger) than the average of its neighbours. Thus perhaps we could say that, in this context anyway, the ∇^2 operator is a sort of "envy operator." In most of the equations encountered in fluid mechanics, the highest order term[7] is a Laplacian term, and it determines the character of the equation.

1.6 Classification of Equations

We will take an informal, schematic approach to this: for more detail see Ames (1977, pp3-5), Patankar (1980, pp 20-22), or texts in mathematics of pde's. The character of a p.d.e. is effectively dictated by the highest order derivatives in the equation, which in fluid mechanics are usually second order spatial derivatives, ie. curvature terms. Recall that a curvature term (eg. on the x axis, $\partial^2\phi/\partial x^2$) couples the dependent variable ϕ to its neighbourhood on both sides, so it implies two-way influence, or two-way spatial connectivity.

"Equilibrium" problems: (Ames, p3) are "problems of steady state in which the equilibrium field of ϕ in a domain D is to be determined by solving the differential equation within D subject to conditions on ϕ on the boundary." These may be thought of as "jury problems," because "the entire solution is passed on by a jury requiring satisfaction of all the boundary conditions and all the internal requirements." The differential equation will be "elliptic" in the spatial variables, ie. will contain the term $\nabla^2\phi$, the Laplacian of the dependent variable[8].

A prototypical elliptic problem is $\nabla^2\phi = 0$ with $\phi = 1$ on the walls. We saw above that the computational molecule for the Laplacian operator ($\nabla^2\phi$) "looks in all directions" and, fancifully

[7] i.e. the term containing the highest order derivative

[8] Classification of the governing equations logically belongs before one embarks into discretized equations; but it is helpful to understand the action of ∇^2 before addressing the classification.

speaking, compares a central value ϕ_C with the average of its neighbours. Intuition may suggest the solution to this problem must be $\phi = 1$ everywhere[9].

Virtually all real problems involving mass, momentum, or energy conservation are governed by differential equations that contain diffusion, ie. equations which contain the Laplacian term. Why? Most such problems admit fluxes (transport) due to molecular motion, ie. diffusion, conduction and viscous drag. The prototypical form of these molecular fluxes is $-K \nabla \phi$, where K [m^2 s^{-1}] is the (appropriate) diffusivity. For reasons that should be apparent from our discussion of the generalized conservation equation, it is the "divergence " $\nabla \cdot (K \nabla\phi) = K \nabla^2\phi$ of such fluxes that appears[10] in the governing equations (here for simplicity I assumed the diffusivity K is constant). So most problems will have a diffusion term operating along each spatial coordinate.

But not infrequently, the diffusive transport term along a particular coordinate is vanishingly small compared to a convective flux[11]. In such a case that coordinate direction is, practically, a "one-way coordinate" (see below), and the diffusion term is insignificant in relation to convection. In this situation conditions *downstream* of a given point P do not affect the tendency/ies at P ($\partial\phi/\partial t$, etc.). A consequence is that no downwind boundary condition is needed. In view of this it is useful to extend our concept of the "ellipticity" of a problem, by assigning that character axis by axis... for due to approximations, a modelling problem may end up having the "elliptic" character on a reduced number of spatial dimensions. Thus we may have ellipticity on only one space axis (eg. our equation retains $\partial^2\phi/\partial x^2$ but is first-order on the other space-axes). In so-called "boundary-layer" flows, for instance, the velocity normal to a boundary (say w, the velocity along z which is normal to a surface) is assumed small or vanishing such that transport along that axis is entirely diffusive (thus the problem is elliptic on the z axis); however the velocity component(s) *parallel* to the surface are not constrained, are therefore large, and eclipse transport by diffusion — so that the alongstream axis is "1-way."

"Marching" or "propagation" problems: "the solution marches out from the initial state guided and modified in transit by the side boundary conditions."

"Time-dependent jury problems": the value of ϕ evolves from some initial value subject to boundary constraints (the equations used in weather prediction are of this form). We have seen that the simplest "computational molecule" for the ∇^2 operator at a given grid point "C" compares the value of ϕ_C with the average of its values at the neighbouring points. In a time-dependent jury problem, an excess in the value of ϕ_C relative to the average of its neighbours will cause a tendency

[9]But don't worry if it doesn't. It may be a useful exercise to solve this on your own, for one space dimension: solve $\partial^2\phi/\partial x^2 = 0$ on $0 \le x \le D$ with $\phi(0) = \phi(D) = 1$.

[10]Here it was assumed K is independent of position; but even if it is not, a Laplacian term must emerge from the flux divergence.

[11]Let's compare advection of water vapour by a $U \sim 20$ m s^{-1} wind with molecular diffusion. The diffusivity of water vapour in air is about $D_w \sim 2$ x 10^{-5} m^2 s^{-1}. If the inhomogeneity of the water vapour field can be expressed as $\Delta\rho_v\, L^{-1}$ where L is a length scale, then the ratio of the convective to diffusive fluxes (called the Peclet number) is $(U\Delta\rho_v)/(D_w\Delta\rho_v L^{-2}) = UL^2/D_w$, a huge number.

for ϕ_C to be reduced at the subsequent time, ie. the ∇^2 operator is a "smoothing" operator. It is associated with diffusion of the property ϕ , ie. the ∇^2 terms normally appear in the equations as the expression of the physical process of diffusion (either molecular diffusion, or, in the case of some models of turbulent convective transfer, turbulent diffusion).

A "**one-way (two-way) coordinate**" is such that conditions at a given location in that coordinate are influenced by changes in conditions on only one side (both sides) of that location. Assuming time travel to be impossible, time is presumably a one-way coordinate. Very commonly, equations involving fluid motion which is close to being parallel to a wall (or the ground) are approximated in such a way that the direction of the flow becomes a one-way coordinate: such problems are solved efficiently by "marching" downstream from a given line along which the solution is known.

1.7 Terminology with respect to errors

Truncation error (Haltiner & Williams, 1980, p 120; Haltiner, 1971, p105; Holton, 2004, p460) "is defined to be the difference between the difference equation and the differential equation." Thus for example if we employ the forward difference for our $u\,\partial\phi/\partial x$ advection term in Eq. 1.9, our truncation error is of order Δx. The difference equation is said to be *consistent* or *compatible* with the differential equation, provided the truncation error vanishes as the grid intervals and the timestep are reduced indefinitely in size.

The exact (but usually unknown) solution to the differential equation is ϕ. Suppose we could obtain an exact solution ϕ_* to the neighbour equations that result from discretization (this is sometimes possible), and let us label the numerical solution ϕ_{Num}. Then:

- *Discretization error* $= \phi - \phi_*$
- *Stability error* $= \phi_* - \phi_{Num}$
- A difference solution ϕ_{Num} is said to be "convergent" if it approaches the true solution ϕ as $\Delta t, \Delta x \to 0$.
- *Lax Equivalence Theorem*: "If a difference equation is consistent with the differential equation it represents then stability is the necessary and sufficient condition for convergence." (JW's comment: presumably there must also be a consistency condition on the numerical initial and boundary conditions too.) As for the meaning of "stability," it is said that "A difference scheme is stable if its solutions remain uniformly bounded functions of the initial state for sufficiently small values of the timestep."

Ordinarily, one will use difference equations that are consistent with the differential equations, and one's numerical scheme will have the property that the numerical solution is convergent. How

small must one push the gridlength's and timestep? Ordinarily one will perform tests with smaller and smaller (Δ_x, Δ_y, Δ_z, Δt) until solutions prove independent of further refinement: one is then entitled to claim the solution is *grid independent*.

Chapter 2

Linear stability analysis of numerical schemes

Sometimes an apparently reasonable discretization of the governing equations and apparently reasonable choice of grid-lengths and timestep, can lead to an "explosion" of the (computed) solution having nothing to do with the reality of the flow.

The linked equations solved in a realistic weather model are (in the case of those equations expressing conservation of momentum) non-linear, and are solved on a non-uniform grid. However quite simple considerations of the "linearised building blocks" of those more-complex equations can suggest necessary conditions for the stability of a numerical procedure. Methods of formal linear stability analysis are well developed.

2.1 Intuitive Stability Analysis

Here and henceforth, let superscript "n" imply time $t = n\Delta t$, while indices I, J denote $x = I\Delta x, y = J\Delta y$ (etc).

2.1.1 Heat equation

A discretization of the 1-d storage+diffusion equation (heat equation) on a uniform grid, using a forward time difference and the simplest computational molecule for the spatial curvature (Truncation Error of $O(\Delta t + \Delta x^2)$) is:

$$\frac{{\phi_I}^{n+1} - {\phi_I}^n}{\Delta t} = \frac{K}{\Delta x^2} \left[{\phi_{I+1}}^n + {\phi_{I-1}}^n - 2{\phi_I}^n\right] \tag{2.1}$$

Now assume that at time n the solution is a "two-gridpoint oscillation in space," ie., ${\phi_I}^n = (-1)^I$, where (please note) I shan't bother to carry the "Num" subscript that reminds us this is a numerical solution.

Now our intuition should tell us that at any later time $n+1, n+2, ...$ the amplitude (of this, or any other wave present at time n) should be *smaller*. Why? Well, here are the pertinent facts.

- this is a problem in which there are no volumetric sources of ϕ
- and it is a linear problem, so there can be no wave-wave interactions that might increase amplitudes of given Fourier components of the solution (this point may be obscure, for now)
- more specifically, it contains only the storage term $\partial\phi/\partial t$ and *diffusion*, and the diffusion term, as we have seen, acts so as to smooth (or damp) the solution progressively as time increases.

Then in view of this intuition, we are entitled to say that if $|\phi_I^{n+1}| > 1$ for any I not only is the wave is growing, but in particular this is unphysical behaviour — ie. there must be some artificial (numerical) instability.

Taking the case where I is ODD so ${\phi_I}^n = -1$, a condition for stability is:

$$\frac{K\Delta t}{\Delta x^2} \leq \frac{1}{2}\,. \tag{2.2}$$

The dimensionless ratio $K\Delta t/\Delta x^2$, which can be thought of as being the reciprocal of a grid Reynolds number $\Delta x\ (\Delta x/\Delta t)/K$ where K is the effective viscosity, is sometimes called the "diffusion number," and the above constraint the "diffusion limit." The conditional instability of this numerical method for the heat equation is confirmed by formal methods.

2.1.2 1D advection equation

A discretization of the 1-d advection equation $\partial_t\phi + U\partial_x\phi = 0$ using centred time and space differences (truncation error of order $\mathrm{O}(\Delta t^2 + \Delta x^2)$) is:

$$\frac{\phi_I^{n+1} - \phi_I^{n-1}}{2\Delta t} = -U\,\frac{\phi_{I+1}^n - \phi_{I-1}^n}{2\,\Delta x} \tag{2.3}$$

This is a "leapfrog scheme," so-called because of the occurrence of ϕ at three times $n-1, n, n+1$. To keep life simple, we'll choose $U = \text{const}$. Now the exact solution for $\phi(x,t)$ is[1]

$$\phi(x,t) = \phi(x - Ut, 0) = \phi_0(x - Ut) \tag{2.4}$$

where $\phi_0(x) = \phi(x,0)$ is the initial field of ϕ. Recall that we know this is the solution, since the differential equation tells us ϕ is constant following the motion. Thus if we were to have an exact numerical solution on the grid it would be expressible as something like

$$\phi_I^n = \phi(I\Delta x, n\Delta t) = \phi_0(I\Delta x - U\ n\ \Delta t) \tag{2.5}$$

Figure (2.1) is a time-space $(t-x)$ diagram which shows how the "domain of influence" upon ϕ_I^n expands out in space and backward in time from the point $n = 4, I = 5$, for an arbitrary choice

[1]We know this because in Lagrangian form ϕ is the solution of $d\phi/dt = 0$, i.e. the field of ϕ is "frozen into the flow" and simply moves with it such that each fluid element keeps its own unchanging value of ϕ.

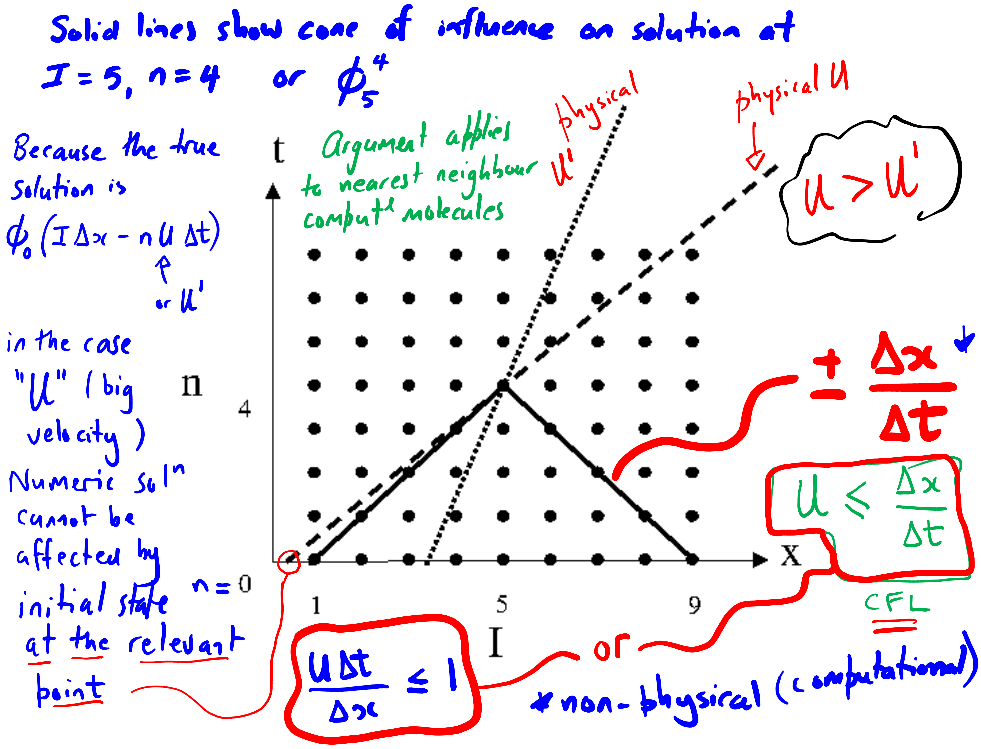

Figure 2.1: Cone of influence; from Haltiner and Williams p121

of $\Delta x, \Delta t$. The slope of the cone of influence is $\Delta x/\Delta t$, which is a velocity defined by our choices of $\Delta x, \Delta t$. Also shown are curves $x = Ut$ drawn upstream from the point of interest, ϕ_5^4, for two possible values of the advection velocity U. The long-dashed line lies outside the domain of influence, hence in that case there is no way for the true solution $\phi_0(x_I - Un\Delta t)$ to influence the numerical solution: the numerical solution cannot be correct. Conversely, the dotted curve, for a smaller U, lies within the domain of influence, hence the correct solution is able to influence the numerical solution. Generalising, the choice of grid spacing and timestep is acceptable provided the path $x = Ut$ lies within the domain of influence. This requires that the Courant number C must obey the restriction:

$$C \equiv \frac{U\ \Delta t}{\Delta x}\ \leq 1\ . \tag{2.6}$$

This is called the "Courant-Friedrichs-Levy" (CFL) condition, which was set out in 1928; again, the CFL condition may be derived formally. According to Lindzen and Fox-Rabinowitz (1989; Monthly Weather Review, Vol. 117), the CFL condition "calls for a time step that is usually much smaller than the time scale associated with the dominant spatial scale of the phenomena being described. The point is that as long as we resolve the smaller scales for which the CFL condition is violated, these scales will eventually be excited by non-linearity or even roundoff error." Also according to Lindzen and Fox-Rabinowitz, "a similar consistency requirement exists between vertical and horizontal resolution.... ; ... excessive horizontal resolution could resolve modes whose vertical wavelength might be too small to be resolved with the existing vertical resolution, and this situation could lead to spatial instability."

2.2 Von Neumann (or Fourier) Linear Stability Analysis

This is a simple method of stability analysis which is applicable only to finite difference schemes on a uniform grid approximating linear equations. The analysis can be applied to a linearised approximation to a non-linear equation to determine a necessary condition for stability. No account is taken of the influence of boundary conditions.

The subject will not be taught this year, but we shall refer to insights that derive from the Von Neumann analysis.

2.3 Computational molecules for the 1-d heat equation

A numerical approximation to the diffusion equation or heat equation (eqn 1.20) using a central time difference and the standard computational molecule for the diffusion term (Richardson's method) is unconditionally unstable (one can prove this using Von Neumann stability analysis).

In the *Dufort-Frankel* method we modify the standard molecule for the curvature (ie. for a component of the Laplacian operator) by replacing $-2\ {T_I}^n$ with $(-{T_I}^{n+1} - {T_i}^{n-1})$. This yields an unconditionally stable, three time level method for the heat equation

$$\frac{{T_I}^{n+1} - {T_I}^{n-1}}{2\Delta t} = K\ \frac{{T_{i+1}}^n + {T_{I-1}}^n - {T_I}^{n+1} - {T_I}^{n-1}}{\Delta x^2} \tag{2.7}$$

which is explicit, because only ${T_I}^{n+1}$ appears as an unknown.

In the *Crank-Nicholson* discretization one takes a forward difference for the time derivative and a linear combination of two molecules for the diffusion term:

$$\begin{aligned} \frac{T_I^{n+1} - T_I^n}{\Delta t} &= \lambda\ K\ \frac{T_{i+1}^{n+1} + T_{I-1}^{n+1} - 2\ T_I^{n+1}}{\Delta x^2} \\ &+ (1-\lambda)\ K\ \frac{T_{i+1}^n + T_{I-1}^n - 2\ T_I^n}{\Delta x^2}\ . \end{aligned} \tag{2.8}$$

If $\lambda = 0$, we have an explicit formula which is conditionally unstable[2]. If $\lambda = 1/2$, we have the scheme of Crank & Nicholson, which is unconditionally stable. This introduces the notion of an implicit scheme: we see that at the new (unknown) time level $(n+1)$ we do not have ${T_I}^{n+1}$ in isolation, but the neighbouring values at $I+1, I-1$ as well — ie. the difference equation for the unknown ${T_I}^{n+1}$ contains two other unknowns as well. Therefore with this formula we cannot, from the earlier-time solution, step forward to obtain the solution at time $n+1$ by a simple, explicit formula. The problem is closed only in the sense that the set of difference equations will contain as many equations as there are unknowns. In this case, solution will necessitate the use of an iterative technique or, more directly, solution of a "tridiagonal" matrix problem — the solution at $(I, n+1)$ is linked to the solution at $(I-1, n+1)$ and $(I+1, n+1)$. More on this later.

[2]Stability criterion?

2.4 Computational molecules for 1-d advection equation

The 1-d advection equation is:

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = 0 \tag{2.9}$$

The *Euler method*

$$\frac{\phi_I^{n+1} - \phi_I^n}{\Delta t} = - U \frac{\phi_{I+1}^n - \phi_{I-1}^n}{2\Delta x} \tag{2.10}$$

uses a forward difference in time, and a central difference in space. It is explicit, and unconditionally unstable.

The *upstream differencing* method uses a forward difference in time, and an upwind difference in space. It is explicit. One expects a damped or neutral solution provided

$$0 \leq \frac{U\, dt}{dx} \leq 1 \tag{2.11}$$

The *trapezoidal implicit* scheme uses a forward difference in time, while for the space derivative the average of the central-difference for two times is taken,

$$\frac{\phi_I^{n+1} - \phi_I^n}{\Delta t} + \frac{U}{2} \left(\frac{\phi_{I+1}^{n+1} - \phi_{I-1}^{n+1}}{2\Delta x} + \frac{\phi_{I+1}^n - \phi_{I-1}^n}{2\Delta x} \right) = 0 \tag{2.12}$$

This scheme is implicit, and unconditionally stable.

The *Euler backward* (or *Matsuno*) scheme is an explicit trial-step method

$$\begin{aligned} \phi_I^{*,n+1} &= \phi_I^n - \frac{U\Delta t}{2\Delta x} \left(\phi_{I+1}^n - \phi_{I-1}^n \right) \\ \phi_I^{n+1} &= \phi_I^n - \frac{U\Delta t}{2\Delta x} \left(\phi_{I+1}^{*,n+1} - \phi_{I-1}^{*,n+1} \right) \end{aligned} \tag{2.13}$$

Here $\phi_I^{*,n+1}$ is a first guess for ϕ_I^{n+1}, which is used in the corrected second step. This scheme is stable so long as $|U\Delta t/\Delta x| \leq 1$. Another explicit 2-step scheme that is subject to the CFL condition is the Lax-Wendroff method (see Haltiner and Williams, p149).

Chapter 3

"Relaxation" method for elliptic differential eqns

Consider the steady-state diffusion equation

$$\frac{\partial T}{\partial t} = \; 0 = \; \kappa \, \nabla^2 T + \; Q \tag{3.1}$$

where (for example) κ is thermal diffusivity and $Q = Q(x, y, z)$ is an arbitrary but steady source. Knowing the action of the Laplacian operator is to smooth out sharp spatial curvature, our intuition suggests the solution should be a field $T(x, y, z, t)$ that is as smooth as can be consistent with the source distribution Q and the boundary conditions, both here unspecified.

It is also a reasonable guess that if we started out with the *wrong* solution and plugged it into the time-dependent equation

$$\frac{\partial T}{\partial t} = R \; = \; \kappa \, \nabla^2 T + \; Q \tag{3.2}$$

then any stable numerical solution technique that is consistent with the differential equation should nudge our solution towards the correct steady state solution... and when we reach that solution the "residual" $R(x, y, z, t)$ will vanish.

Let us specialize to 2 space dimensions, and discretize as

$$\kappa \, \frac{T^n_{i+1,j} + T^n_{i-1,j} - 2T^n_{i,j}}{\Delta x^2} + \; \kappa \, \frac{T^n_{i,j+1} + T^n_{i,j-1} - 2T^n_{i,j}}{\Delta y^2} + \; Q_{i,j} = \; R^n_{i,j} \tag{3.3}$$

where $T^n_{i,j}$ is the n^{th} trial field (ie. not the correct solution; if $n = 1$ it is our first guess field). Now, any non-zero value of the residual R^n_{ij} indicates that we don't yet have a valid solution, at least at this grid point. However we can adjust the value of T at the local gridpoint to make the (new) residual exactly zero by making an adjustment

$$T^{n+1}_{i,j} \leftarrow \; T^n_{i,j} + \; \alpha \, R^n_{i,j} \tag{3.4}$$

Substituting this value into Eqn. (3.3) we get

$$\begin{aligned} \kappa \, \frac{T^n_{i+1,j} + T^n_{i-1,j} - 2\left(T^n_{i,j} + \alpha R^n_{i,j}\right)}{\Delta x^2} \quad &+ \quad \kappa \, \frac{T^n_{i,j+1} + T^n_{i,j-1} - 2\left(T^n_{i,j} + \alpha R^n_{i,j}\right)}{\Delta y^2} + \, Q_{i,j} \\ &= \quad R^n_{i,j} - \, 2 \, \kappa \, \alpha \, R^n_{i,j} \, \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \\ &= \quad R^{n+1}_{i,j} \, . \end{aligned} \tag{3.5}$$

But we may choose α to make $R^{n+1}_{i,j}$ *vanish*, viz:

$$\alpha = \, \frac{1}{2\,\kappa} \, \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right)^{-1} \equiv \frac{1}{2\,\kappa} \, \frac{\Delta x^2 \, \Delta y^2}{\Delta x^2 + \, \Delta y^2} \tag{3.6}$$

Of course, the moment we change $T_{i,j}$ we affect the residual not only locally but at neighbouring points. However it turns out that if one simply iterates across the whole grid, repeatedly, in any order, the residuals progressively get smaller as the iteration count n increases.

One needs a criterion for cessation of iterations. How small must the residual $R_{i,j}$ be driven? A typical approach is to demand that the root mean square residual

$$\sigma_R = \, \left(\frac{1}{I_{mx} \, J_{mx}} \, \sum_{i=1}^{Imx} \, \sum_{j=1}^{Jmx} \, R^2_{i,j} \right)^{1/2} \tag{3.7}$$

be smaller than some criterion ϵ, where ϵ is chosen on physical grounds.

Finally, note that other methods can be applied to solve this type of problem. A popular approach is the Alternating Direction Implicit (ADI) scheme. Here one sets up an implicit discretization, then makes a sequence of alternating sweeps down the (two or three) axes, until the residuals are judged satisfactorily small.

Chapter 4

Aliasing and Non-Linear Computational Instability

4.1 Aliasing

For the moment, let us conceive of wishing to store a record of some variable, say ϕ, along a line: say, $\phi(x)$ on the interval $-L/2 \leq x \leq L/2$. That's an infinite amount of information. We'll need a universe of 1 TB drives! But suppose we envisage that signal or record as being the superposition of a large number of sine and cosine waves, each with arbitrary magnitude and phase? In doing so, we would be envisaging a "Fourier decomposition" of $\phi(x)$: permitted an infinite number of sine and cosine waves, we can fit that darn curve to an arbitrary level of accuracy. OK, let's do that. We represent a curve in terms of a superposition of (a summing of) waves (or wave contributions).

A grid with spacing Δx can represent only waves having wavelength[1]

$$\lambda \geq \lambda_{min} = \ 2\Delta x \tag{4.1}$$

or in terms of wavenumber $k = 2\pi/\lambda$

$$k \leq k_{max} = \ \pi/\Delta x \ . \tag{4.2}$$

To see that this is so, multiply onto a grid $x(I) = I\Delta x$ the wave $\phi = \sin\left[k_{max}(1+\epsilon)x\right]$ where $0 < \epsilon \ll 1$ (this curve is plotted on Figure 4.1). The values of ϕ at the grid points are[2]

$$\phi(I) = \sin\left[\frac{\pi}{\Delta x}\ (1+\epsilon)\ I\ \Delta x\right] = \ \sin\left[\pi I + \ \pi I\epsilon\right] = \ (-1)^I\ \sin\left[\pi I\epsilon\right] \tag{4.3}$$

Table (4.1) lists the values of the function (c.f. Fig 4.1), and in comparison, the values of $\phi^* = \sin[k_{max}(1-\epsilon)x]$. Since $\epsilon << 1$ the factor $\sin(\pi\epsilon I)$ varies slowly: but our $\phi(I)$ oscillates with I; it is a two-gridpoint wave. The wave which "actually" has wavenumber $k_{max}(1+\epsilon)$ and so is larger than k_{max} has been *aliased* as a longer wave with wavenumber $k_{max}(1-\epsilon)$. In general, a wave at $k = k_{max} + \delta k$ (where $\delta k < k_{max}$) is represented on the discrete grid as a wave of lower wavenumber $k^* = 2k_{max} - k = k_{max} - \delta k$.

[1] This is related to, or indeed equivalent to, Shannon's sampling theorem in Information Theory.

[2] $\sin(A+B) = \sin A\ \cos B + \ \cos A\ \sin B$, and let $A = \pi I$...

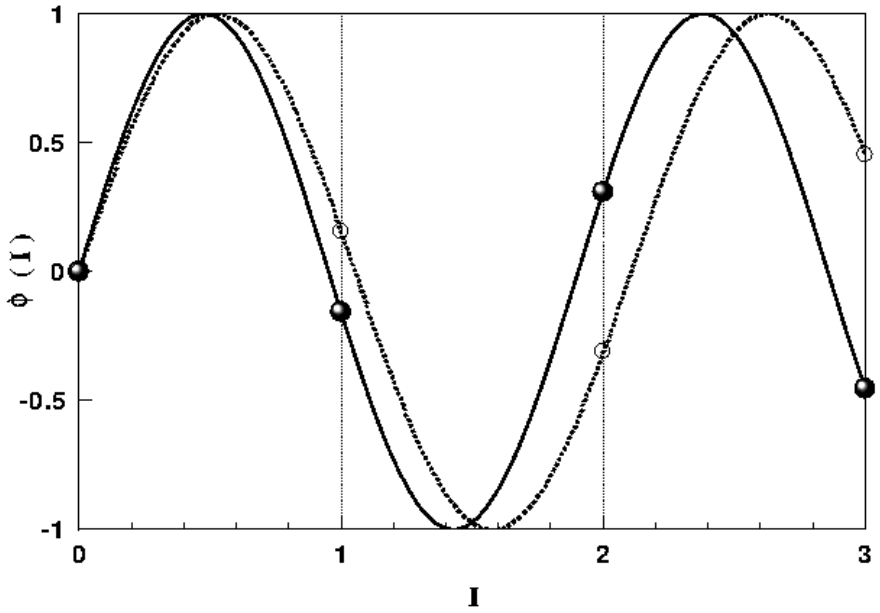

Figure 4.1: Illustrating the function $\phi = \sin[\pi(1+0.05)I]$ (solid curve) and its representation on the discrete grid (symbols •). The dashed line is the curve $\sin[\pi(1-0.05)I]$ and symbols (∘) show its representation. All information along the curve(s) is lost — hidden — and so effectively, to within a sign change, our 'k^{+}_{max}' curve has the same representation as the (slower) 'k^{-}_{max}' curve.

4.2 Non-linear Computational Instability

When a finite differencing method is applied to the non-linear advection equation

$$\frac{\partial u}{\partial t} + \; u \, \frac{\partial u}{\partial x} = 0 \tag{4.4}$$

the CFL restriction, that $u\Delta t/\Delta x \leq 1$, is a necessary but not a sufficient condition to ensure computational stability.

Why? Let us assume that at time n our discrete solution on the grid $x = I\Delta x$ is a wave of (real) wavenumber k and amplitude U^n,

$$u^n_I = \; U^n \; e^{j\,k\,(I\Delta x)} \; . \tag{4.5}$$

Then if we use a central difference for $\partial u/\partial x$

$$u \, \frac{\partial u}{\partial x} = \; (U^n)^2 \; e^{j\,k\,I\,\Delta x} \; \frac{e^{j\,k\,(I+1)\,\Delta x} - \; e^{j\,k\,(I-1)\,\Delta x}}{2\Delta x} \tag{4.6}$$

and this is readily simplified to

$$u \, \frac{\partial u}{\partial x} = \; \frac{j \; (U^n)^2}{\Delta x} \; \sin(k\Delta x) \; e^{j\,2k\,I\Delta x} \; . \tag{4.7}$$

Now if we use a forward difference in time (results would be similar using a leapfrog scheme) we have:

$$u_I^{n+1} = u_I^n + \frac{\partial u}{\partial t}\,\Delta t = u_I^n - u\,\frac{\partial u}{\partial x}\,\Delta t \tag{4.8}$$

which yields:

$$u_I^{n+1} = U^n\, e^{j\,k\,I\Delta x} - j\frac{\Delta t}{\Delta x}\,(U^n)^2\,\sin(k\Delta x)\,\left[\cos(2kI\Delta x) + j\,\sin(2kI\Delta x)\right]\,. \tag{4.9}$$

So u_I^{n+1} has a real component

$$U^n\,\left[\frac{\Delta t}{\Delta x}\,U^n\,\sin(k\Delta x)\right]\,\sin(2k\,I\Delta x) \tag{4.10}$$

which is a wave at double the starting wavenumber. To be sure, it may only have small amplitude

$$U^n\,\left[\frac{\Delta t}{\Delta x}\,U^n\,\sin(k\Delta x)\right] \tag{4.11}$$

since we presumably impose the CFL condition. But every step will accentuate the problem.

What's happening? A wave-wave interaction in the non-linear term has lead to the production of a shorter wave (higher wavenumber). In our special case it is in fact an interaction of waves of the same wavenumber, but you can appreciate that if u_I^n is some more complex superposition of waves, say (adding only minimal complexity for starters)

$$u_I^n = U_1^n\,\sin(k_1\,I\Delta x) + U_2^n\,\sin(k_2\,I\Delta x) \tag{4.12}$$

then $u\partial u/\partial x$ will produce interactions of form

$$\sin(k_1 I\Delta x)\cos(k_2 I\Delta x) = \frac{1}{2}\,\left[\sin((k_1+k_2)\,I\Delta x) + \sin((k_1-k_2)\,I\Delta x)\right]\,. \tag{4.13}$$

Here, starting with waves k_1 and k_2, we have gathered a longer wave $k_1 - k_2$ and a shorter wave $k_1 + k_2$. For a more-general superposition as the starting point, $u\,\partial u/\partial x$ will contain many wave cross-products.

We are gathering power at higher wavenumber (steepening gradients). Now this is physically proper: that is what should be happening (in the real meteorological equations other terms would moderate this behaviour). But what if, as this progresses, we generate power at wavenumbers too high to be represented on the grid, ie. at $k > \pi/\Delta x$? For example, if we start with our "pure" wave k taking the value $k = k_{max}\,(1+\epsilon)\,/2$ we come out one timestep later with wave

$$\sin(2kI\Delta x) = \sin(\pi I + \pi\epsilon I) = (-1)^I\,\sin(\pi\epsilon I) \tag{4.14}$$

which due to aliasing, instead of being represented as it should be, a wave shorter than k_{max}, has appeared as a wave just long enough to be represented on the grid.

NLCI of a finite difference scheme, then, is the following: the non-linear term(s), such as $u\,\partial u/\partial x$, cause (and correctly so!) wave-wave interactions that produce shorter wave components; the problem is that when these are too short for the grid, they are aliased, so that the finite difference method falsely accumulates energy in the highest wavenumbers representable on the grid.

4.2.1 Getting around NLCI

Possibilities are:

1. Use a finite difference scheme that damps short waves. This is sometimes achieved by including a bogus smoothing (diffusion) term ∇^2. Unless this fictitiously added term has a typical magnitude that is small compared to the real terms, one has thereby changed the dynamics of the flow model. In turbulent flow such a diffusion term is present anyway, the expression of a turbulent diffusion process; and that may be sufficient to control the gradient-sharpening action of the non-linear advection term.

2. Use a spectral method of discretization, rather than finite differencing. Here variation along the axes (say x) is restricted to be of known, analytical form (polynomial or trigonometric curves) and, in a sense, there is no grid — discretisation takes place in wavenumber space rather than physical space. Thus if we include only the truncated set of wavenumbers $k_1, k_2, ...k_N$ (where k_N is the shortest wave included) then the non-linear interaction of k_{N-1} and k_N to produce $k_{SUM} = k_{N-1} + k_N > k_N$ is not problematical, because k_{SUM} lies outside the truncated set of included waves and is entirely ignored. No aliasing occurs. This is one of the advantages of "spectral" models.

Table 4.1: Example of the aliasing of an exact function $\phi(x)$ when represented on a discrete grid $x(I) = I\Delta x$, for which the finite interval Δx entails that the maximum representable wavenumber will be $k_{max} = \pi/\Delta x$). Here the function represented on the grid is $\phi = \sin[k_{max}(1+\epsilon)x]$ with $\epsilon = 0.05$. Also listed (for comparison) are values of $\phi^* = \sin[k_{max}(1-\epsilon)x]$.

I	$\pi\epsilon I$	$\phi(I)$	$\phi^*(I)$
0	0	0	0
1	0.05 π	-0.156	0.156
2	0.10 π	+0.309	-0.309
3	0.15 π	-0.454	+0.454
4	0.20 π	+0.588	-0.588
5	0.25 π	-0.707	+0.707
6	0.30 π	+0.809	-0.809
7	0.35 π	-0.891	+0.891
8	0.40 π	+0.951	-0.951
9	0.45 π	-0.988	+0.988
10	$\pi/2$	+1.000	-1.000
11	0.55 π	-0.988	+0.988
12	0.60 π	+0.951	-0.951
13	0.65 π	-0.891	+0.891

Chapter 5

Dimensional Analysis

Dimensional analysis is a tool of great power in the examination of any physical system (Bridgman 1922; Langhaar 1980; Panton 2013). It can guide not only the theoretician, but also the design and analysis of experiments. According to Drazin & Reid (2004) the use of dimensional analysis in fluid mechanics was encouraged by Reynolds' discovery that stability (sustained plane-parallel, or laminar, motion) of Poiseuille flow in a pipe depends only on the dimensionless "Reynolds number" $R_e = Ud/\nu$ (U=velocity, d=diameter, ν = kinematic viscosity of air).

It is obvious, though we may never have remarked the fact, that physical laws must be dimensionally homogeneous. If $A = B + C$ is a meaningful law of a sensible physical system, then the dimensions $[A], [B], [C]$ of the terms must be equal: a valid equation cannot mix (additively) terms with units of kilograms and terms with units of seconds.

As a consequence of this commonplace, the Buckingham Pi Theorem states that:

> If an equation in n variables is dimensionally homogeneous with respect to m fundamental dimensions it can be expressed as a relation between $n - m$ independent dimensionless groups. (Buckingham 1914)

5.1 Example: Pendulum

Suppose an observer suspects or hypothesizes that the period T of a simple pendulum is controlled by its mass M, length L, and gravity g (see Fig. 5.1).

The number of variables involved is $n = 4$ and we have $m = 3$ fundamental dimensions[1]. Thus the governing equation must involve only $n - m = 1$ non-dimensional variables.

Therefore s/he should seek an equation of form $N = const.$, where N is dimensionless. Only a single non-dimensional number can be formed from T, M, L, g, namely $\frac{T}{\sqrt{L/g}}$, so $\frac{T}{\sqrt{L/g}} = \text{const.}$

[1]Time t has not been counted by this astute thinker, for he stipulates the process under investigation is a non-decaying, purely oscillatory motion, and does not ask to know any characteristic of the motion that *might* evolve in time.

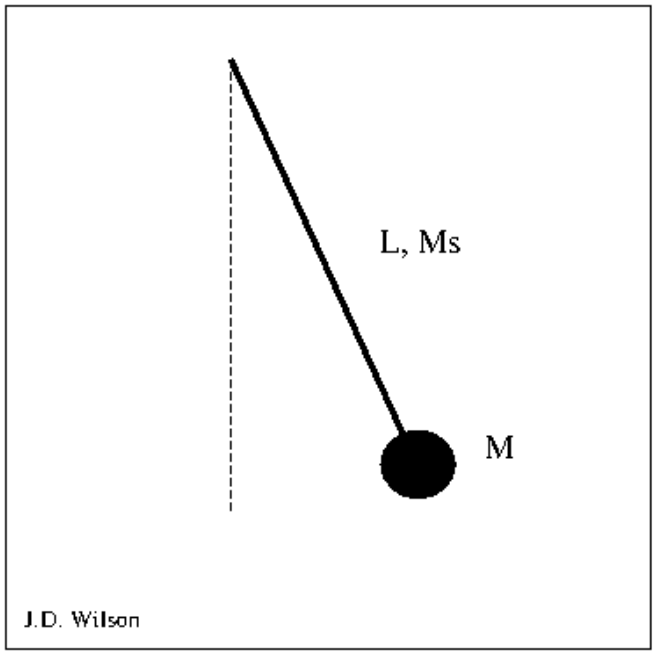

Figure 5.1: The period (T) of a pendulum in a gravitational field (g, $\mathrm{m\,s^{-2}}$) is suspected to depend on the masses (M, M_s) and the length (L) of the string.

We say that the "controlling timescale" is (L/g). The validity of this prediction is confirmed by a force balance, which yields for small amplitude motion:

$$\frac{T}{\sqrt{L/g}} = 2\pi \; . \tag{5.1}$$

The mass M has been found to be irrelevant. The inclusion of irrelevant variables adds work but does not invalidate the procedure. The omission of a truly relevant variable will yield a relationship that does not agree with reality.

If we add the possibility that the mass of the string M_s may be important, we now have $n = 5, m = 3$. The prediction is then:

$$\frac{T}{\sqrt{L/g}} = F_1\left(\frac{M}{M_s}\right); , \tag{5.2}$$

and we know that $F_1 \to 2\pi$ as $M/M_s \to \infty$. A further factor neglected to this point is frictional damping due to the relative motion of the pendulum and the fluid in which it is immersed. The kinematic viscosity $\nu\,[\mathrm{m^2\,s^{-1}}]$ can be added to the analysis[2], and we note that $L^{3/2}g^{1/2}$ has the same units as ν. Therefore we can write

$$\frac{T}{\sqrt{L/g}} = F_2\left(\frac{M}{M_s}, \frac{\nu}{L^{3/2}g^{1/2}}\right); , \tag{5.3}$$

[2]For air at 20°C and standard pressure, $\nu \approx 1.55 \times 10^{-6}\ \mathrm{m^2\,s^{-1}}$.

and we know that $F_2 \to 2\pi$ as $M/M_s \to \infty$ and $\nu \to 0$.

Often, finding the non-dimensional variables is easy. In case not, there are straightforward mechanical ways of finding the non-dimensional ratios given the variables involved.

5.2 Method of Indices

(Schlicting, p15; or Section 9.3.2 Arya).

An investigator feels that the drag force F_D per unit of cross-stream length on a cylinder in a steady fluid stream depends only on the speed U, the cylinder diameter d, the density ρ, and the dynamic viscosity μ.

Assume the dependent variable F_D may be written in the form of a series of terms, each of which is a dimensionally correct product of independent variables:

$F_D = \sum_i c_i U^{\alpha_i} d^{\beta_i} \rho^{\gamma_i} \mu^{\delta_i}$

where the c_i are dimensionless constants. Dimensional homogeneity requires that:

$F_D = U^{\alpha_i} d^{\beta_i} \rho^{\gamma_i} \mu^{\delta_i}$

which implies that:

$$\begin{aligned} kg: \quad 1 &= \gamma_i + \delta_i \\ m: \quad 0 &= \alpha_i + \beta_i - 3\gamma_i - \delta_i \\ s: \quad -2 &= -\alpha_i - \delta_i \end{aligned} \tag{5.4}$$

Here we have 3 equations in 4 unknowns, so 3 unknowns may be expressed in terms of the other 1, and the i^{th} term may be written:

$$c_i U^{2-\delta_i}\ d^{1-\delta_i}\ \rho^{1-\delta_i}\ \mu^{\delta_i} = c_i\ U^2 \rho d \left(\frac{\mu}{\rho U d}\right)^{\delta_i} \tag{5.5}$$

Thus:

$$\frac{F_D}{\rho U^2 d} = \sum_i c_i \left(\frac{Ud}{\nu}\right)^{-\delta_i} = F\left(\frac{Ud}{\nu}\right). \tag{5.6}$$

The drag, made dimensionless by $\rho U^2 d$, is a function only of the Reynolds number (the dynamic viscosity μ and the kinematic viscosity $\nu, [m^2 s^{-1}]$ are related by $\nu = \mu/\rho$). This suggests the investigator plot his observations in the form $F_D/(\rho U^2 d)$ versus Ud/ν in the hope of finding a universal relationship - a very valuable clue. (Check for yourselves that the correct number of dimensionless ratios have arisen, ie. our result is consistent with the Pi theorem.)

Selection of the relevant parameters is the critical step. The process may rely on intuition, on limited experimental data, or, most mechanically but most reliably, on a knowledge of the governing equations and their boundary and initial conditions (even if the latter cannot be solved).

5.3 Example: Laminar pipe flow

Let a pipe of radius R lie along the x-axis ((see Fig. 5.2)). The fluid has constant density ρ and kinematic viscosity ν. The only non-zero velocity component is u (along the pipe), which must vanish at the walls of the pipe. The u-momentum equation (1 component of the Navier-Stokes equations) is:

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = \frac{-1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{5.7}$$

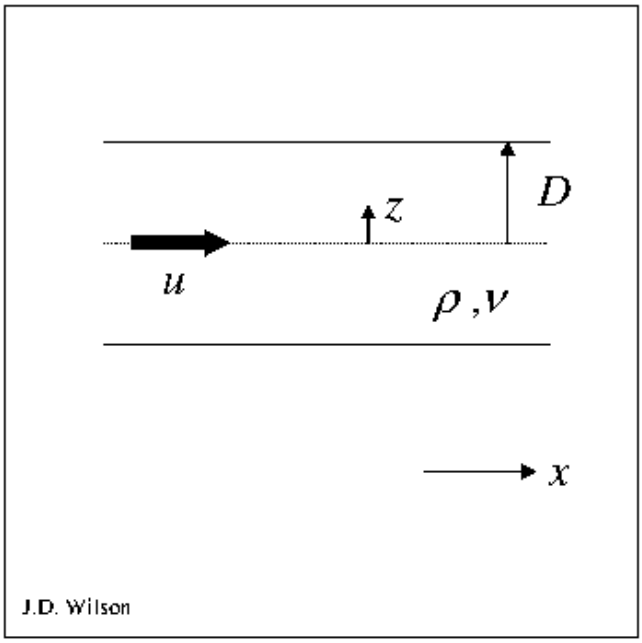

Figure 5.2: Laminar flow in a pipe along the x-axis. How does velocity $u(r)$ vary with radius r?

Assume steady flow, with velocity u independent of x (a necessary consequence of the continuity equation: prove that for yourselves). Several terms, including the local time derivative, vanish. We can extract from the equation and boundary conditions the following variables.

- Variables of the fluid: ρ, ν
- Variables of the flow: $\frac{\partial p}{\partial x}$ (the "forcing")
- Geometric variables: R (arising from the b/cond)
- Location variables: r
- Dependent variable of interest u

Now we have $n = 6, m = 3$ (fundamental dimensions are length, time mass). Thus we seek a relationship between $n - m = 3$ non-dimensional ratios. These ratios can be found by inspection (or mechanically by the method of indices if needbe) to be:

$$\frac{r}{R}\ ,\ \frac{ur}{\nu}\ ,\ \frac{R^3}{\nu^2\rho}\frac{\partial p}{\partial x} \tag{5.8}$$

Hence:

$$\frac{ur}{\nu} = F\left(\frac{R^3}{\nu^2\rho}\frac{\partial p}{\partial x}, \frac{r}{R}\right) \tag{5.9}$$

Note also that we can find other non-dimensional ratios, such as $ur/\nu * R/r$: but these are not independent of the chosen three. The chosen three are not unique, but there can be only three and they must be independent.

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