EAS572, "The Atmospheric Boundary Layer" <u>Mid-term Exam</u> 19 Oct., 2010

<u>Professor</u>: J.D. Wilson <u>Time available</u>: 80 mins <u>Value</u>: 15%

Note: Indices $j = (1, 2, 3)$ are to be interpreted as denoting respectively the (x, y, z) components, e.g. $\vec{u} \equiv u_j \equiv (u_1, u_2, u_3) \equiv (u, v, w)$. In equations or symbols the summation convention applies for repeated alphabetic subscripts (e.g. $u_j u_j$). Symbols $p_R, \rho_R, T_R, \theta_R$ represent pressure, density, temperature and potential temperature of the reference state.

Multichoice ($10 \times \frac{1}{2}\% = 5\%$)

1. In Reynolds' equation

$$\frac{\partial \overline{u}_1}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_1}{\partial x_j} = -\frac{1}{\rho_R}\frac{\partial \overline{p}}{\partial x_1} - \frac{\partial \overline{u'_j u'_1}}{\partial x_j} + \nu \nabla^2 \overline{u}_1 \qquad (1)$$

for the mean streamwise velocity (here symbolized $\overline{u}_1$ and running along the x_1-axis), "turbulent friction" is identifiable as the term

(a) $\nu \nabla^2 \overline{u}_1$
(b) $\overline{u}_j \, \partial \overline{u}_1 / \partial x_j$
(c) $-\partial \, \overline{u'_j u'_1} / \partial x_j$ ✓✓
(d) $\partial \overline{u}_1 / \partial t$

2. Again referring to Eqn. (1), the quantity $\overline{u}_j \, \partial \overline{u}_1 / \partial x_j$ expands as

(a) $\overline{u}_1 \, \partial \overline{u}_1 / \partial x_j \, \delta_{1j}$
(b) $\overline{u}_1 \, \partial \overline{u}_1 / \partial x_1 + \overline{u}_2 \, \partial \overline{u}_1 / \partial x_2 + \overline{u}_3 \, \partial \overline{u}_1 / \partial x_3$ ✓✓
(c) $\overline{u}_1 \, \partial \overline{u}_1 / \partial x_1 + \overline{u}_2 \, \partial \overline{u}_2 / \partial x_2 + \overline{u}_3 \, \partial \overline{u}_3 / \partial x_3$
(d) $\overline{u}_1 \, \partial \overline{u}_j / \partial x_j$

3. Assuming the velocity field u_j is non-divergent. Which of the following expressions for the term $u_j \, \partial \phi / \partial x_j$ (where ϕ is an arbitrary scalar) is **not** correct?

(a) $u \, \partial \phi / \partial x + v \, \partial \phi / \partial y + w \, \partial \phi / \partial z$
(b) $\partial \, u\phi / \partial x + \partial \, v\phi / \partial y + \partial \, w\phi / \partial z$
(c) $\vec{u} \, \nabla \cdot \phi$ ✓✓
(d) $\vec{u} \cdot \nabla \phi$
(e) $\nabla \cdot (\vec{u} \, \phi)$

4. Suppose signals p, q have zero mean (i.e. $\overline{p} = \overline{q} = 0$) whereas the mean of a third signal r is arbitrary (may be non-zero). Reynolds averaging the product $p\,q\,r$ gives for $\overline{p\,q\,r}$ the value:

   (a) $\overline{p'q'r'}$

   (b) $\overline{p'q'}\;\overline{r}$

   (c) $\overline{p'q'}\;\overline{r} + \overline{p'q'r'}$ ✓✓

   (d) 0

   (e) $\overline{r}$

5. Under the Boussinesq approximation the gravitational acceleration $(-g)$ in the vertical momentum equation

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \ldots - g$$

   (here showing only relevant terms) is replaced by a "reduced gravity" term, $+g\tilde{T}/T_R$. Here $\tilde{T}$ is the temperature *deviation* from a hydrostatic and adiabatic reference state (i.e. a reference state whose pressure $p_R(z)$ obeys the hydrostatic law, whose potential temperature is constant and equal to θ_R, and whose true temperature T_R varies with height at the adiabatic lapse rate), such that total temperature $T = T_R + \tilde{T}$ and total pressure $p = p_R + \tilde{p}$.

   Along with the modified gravity term, the vertical pressure gradient force $-\rho^{-1}\,\partial p/\partial z$ is replaced with

   (a) $-\rho_R^{-1}\,\partial p/\partial z$

   (b) $-\rho^{-1}\,\partial \tilde{p}/\partial z$

   (c) $-\rho_R^{-1}\,\partial \tilde{p}/\partial z$ ✓✓

   (d) $-\,\nabla \cdot (p_R/\rho_R)$

   (e) 0

6. One important consequence of the Boussinesq approximation is that the velocity field $\vec{u} \equiv u_j$ satisfies

   (a) $\nabla \cdot \vec{u} = 1$

   (b) $\nabla \cdot \vec{u} = 0$ ✓✓

   (c) $\nabla \times \vec{u} = 0$

   (d) $u_j\,\partial u_i/\partial x_j = g\,\delta_{i3}$

   (e) $\vec{u} = (u, 0, 0)$ where $u = u(x, y, z, t)$

7. The power spectral density $S_{ww}(f)$ of the vertical velocity fluctuation is *defined* such that

$$\sigma_w^2 = \int_{\ln f=-\infty}^{\infty} f\, S_{ww}(f)\, d\ln f\ ,$$

where σ_w^2 is the variance. Let f_{mx} label the frequency at which $f\, S_{ww}(f)$ is a maximum. Referring specifically to height z within a neutrally stratified surface layer, the time scale $1/f_{mx}$ characterizes the ____ range of eddies, and should be (roughly) proportional to ____

(a) energy-containing; $u_*\, z^{-1}$
(b) dissipation (smallest and fastest); $u_*\, \ln z$
(c) energy-containing; $z\, u_*^{-1}$ ✓✓
(d) dissipation (smallest and fastest); $u_*\, z^{-1}$

8. Cauchy's equation of motion for an arbitary real fluid reads

$$\rho\, \frac{du_i}{dt} = \rho\, F_i + \frac{\partial \tau_{ij}}{\partial x_j}\ . \tag{2}$$

Here $\partial\tau_{ij}/\partial x_j$ represents ____ and may be named ____

(a) acceleration due to body forces; the divergence of the stress tensor
(b) acceleration due to surface (contact) forces; the stress tensor
(c) acceleration due to body forces; the stress tensor
(d) acceleration due to surface (contact) forces; the divergence of the stress tensor ✓✓

9. Comparing Eqn. (2) with Reynolds' Eqn. (1) we note the absence of any explicit pressure gradient term in Cauchy's equation. Thus the diagonal elements $(\tau_{11}, \tau_{22}, \tau_{33})$ of his tensor τ_{ij} must be ____

(a) $(p,\ p,\ p)$
(b) $(p/\rho,\ p/\rho,\ p/\rho)$
(c) $(-p,\ -p,\ -p)$ ✓✓
(d) $(u_1,\ u_2,\ u_3)$

where p is the pressure.

10. Neglecting buoyancy and Coriolis terms, the transport equation for vorticity ω_j reads

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 \omega_i \ .$$

If there were such a physical thing as “inviscid, two-dimensional flow,” both terms on the r.h.s. would vanish. In such a flow

(a) vorticity is conserved following the motion ($d\omega_i/dt = 0$) ✓✓

(b) $\nabla \cdot \vec{u} = 0$

(c) $\partial \omega_i / \partial t = 0$

(d) $\vec{\omega} \equiv \nabla \times \vec{u} = 0$

(e) $\partial \omega_i / \partial x_j = 0$

1 Short answer ($2 \times 5\% = 10\%$)

Instructions: Please answer any **two** of the following questions. Use diagrams wherever they may be helpful. Be sure to state any assumptions or simplifications you make. ***No page limit applies.***
23 Oct. 2010: Instructor's skeleton answers/marking criteria provided at back.

1. Explain the Monin-Obukhov similarity theory (MOST) of the atmospheric surface layer. Give the context within which one turns to a similarity theory rather than a rigorous theory based on conservation pricriples. Define the limited conditions under which MOST claims to apply. Delineate the vertical extent of the layer where it is valid; state which quantities MOST posits as "controlling" the flow statistics, relating the choice to the notion of the surface layer as a "constant flux layer." Explain the significance of the Obukhov length and the meaning of the limit $|z/L| \to 0$. Assuming the MO universal function $\phi_m(z/L) \equiv (k_v z/u_*)\partial \overline{u}/\partial z$ for the dimensionless mean wind shear in stable stratification is $\phi_m = 1+5z/L$, deduce by integration a formula for the mean wind profile $\overline{u}(z)$.

2. Explain the *energetics* of turbulence in a stationary and horizontally-homogeneous atmospheric surface layer. Your discussion should make reference to the budget equations (given as data) for the turbulent kinetic energy k, and for the variances (σ_u^2, σ_v^2, σ_w^2) whose sum is $2k$. Support your discussion by a diagrammatic representation of the energetics, and make pertinent reference to the flux Richardson number and the Obukhov length.

3. The generalized conservation equation derived in class is

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \vec{F}_\phi + Q\ .$$

   Assume $\phi = \rho_R\, c_p\, \theta$ where θ is the potential temperature, i.e. ϕ is to be taken as the concentration of sensible heat ($\mathrm{J\,m^{-3}}$) with the product $\rho_R c_p$ (reference density $\times$ specific heat capacity) treated as a constant. By appropriately specifying the sensible heat flux vector, then applying Reynolds averaging, and subsequently discarding terms where justifiable, derive a conservation equation for the mean potential temperature $\overline{\theta}(z)$ in the horizontally-homogeneous ABL. Please note (and justify) all simplifications, restrictions and assumptions you make.

   From an atmospheric modelling perspective, your heat equation would need boundary conditions on ground and at the top of the ABL (i.e. at $z = 0, \delta$). So called "flux boundary conditions" specify the kinematic heat flux density $\overline{w'\theta'}$ (or equivalently $\rho_R c_p\ \overline{w'\theta'}$) at those levels. Explain how this couples your equation to the surface energy budget (given as data).

Data

Note: Below ρ_R, T_R are the reference density and reference (Kelvin) temperature.

- The turbulent kinetic energy (TKE, "k") equation, assuming steady state and horizontal uniformity, neglecting viscous transport ($\nu\nabla^2 k$), and assuming a uni-directional mean flow aligned with the x-axis$(\overline{u}, 0, 0)$, is:

$$\frac{\partial k}{\partial t} = 0 = -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} + \frac{g}{T_R}\,\overline{w'T'} - \frac{\partial}{\partial z}\,\overline{w'\;(p'/\rho_R + \;e'\;)} \;-\; \epsilon \;,$$

where $e' \equiv \left(u'^2 + v'^2 + w'^2\right)/2$ and ϵ is the TKE dissipation rate.

- The corresponding budget equation for $\overline{u'^2}$:

$$\frac{\partial \overline{u'^2}}{\partial t} = 0 = -2\,\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} \;-\; \frac{\partial \overline{w'u'^2}}{\partial z} \;+\; \frac{2}{\rho_R}\,\overline{p'\,\frac{\partial u'}{\partial x}} - \epsilon_{uu}\;, \tag{3}$$

where it is common to adopt the assumption of local isotropy and write $\epsilon_{uu} \equiv \epsilon_{vv} \equiv \epsilon_{ww} \equiv (2/3)\,\epsilon$

- The corresponding budget equation for $\overline{v'^2}$:

$$\frac{\partial \overline{v'^2}}{\partial t} = 0 = -\;\frac{\partial \overline{w'v'^2}}{\partial z} \;+\; \frac{2}{\rho_R}\,\overline{p'\,\frac{\partial v'}{\partial y}} - \epsilon_{vv} \tag{4}$$

- The corresponding budget equation for $\overline{w'^2}$:

$$\frac{\partial \overline{w'^2}}{\partial t} = 0 = +2\,\frac{g}{T_R}\,\overline{w'T'} - \frac{\partial}{\partial z}\left(\overline{w'^3} + \frac{2}{\rho_R}\,\overline{p'w'}\right) \;+\; \frac{2}{\rho_R}\,\overline{p'\,\frac{\partial w'}{\partial z}} - \epsilon_{ww} \tag{5}$$

- The flux Richardson number

$$R_i{}^f = \frac{\frac{g}{T_R}\,\overline{w'T'}}{\overline{u'w'}\;\partial\overline{u}/\partial z + \;\overline{v'w'}\;\partial\overline{v}/\partial z} \tag{6}$$

- The Obukhov length $L = -\,u_*^3\,T_R\,\left(k_v\,g\,\overline{w'T'}\right)^{-1}$. In unstable stratification $R_i^f \approx z/L$

- The 'surface energy balance' on a reference plane at the base of the atmosphere is expressed by the equation

$$Q^* \;=\; Q_H \;+ Q_E \;+ Q_G$$

where all fluxes are in [W m^{-2}]. Sign convention: Q^* the net radiation, positive if directed towards the surface; Q_H, Q_E the sensible and latent heat fluxes, positive if directed from the surface towards the atmosphere; Q_G the heat flux to storage beneath the reference plane, positive if directed from the surface into crop/forest/ground/lake/ocean.

Skeleton short answers/marking criteria

1. MOST

To obtain full marks, it was necessary to make each of the following points:

[0.5] Context: unavailability of any rigorous theory, due to the closure problem

[0.5] Claims to apply in horizontally-homogeneous conditions

[0.5] And in a layer $z_0 \ll z \ll \delta$ or equivalently, within the surface layer (depth order $\delta/10$) but above the roughness sublayer

[1.0] posits the state of this layer controlled by kinematic fluxes of momentum and heat ($\overline{u'w'}$, $\overline{w'T'}$); a buoyancy parameter g/T_0; and the coordinate z

[0.5] kinematic fluxes of momentum and heat considered effectively constant across the layer

[0.5] L explained as relating to stratification; $L < 0$ ($L > 0$) in unstable (stable) stratification, $z/L \to 0$ the neutral limit

[0.5] $z/|L| < 1$ defining the layer in which shear production is more important than buoyant production of TKE

[0.5] integrate the wind shear to get

$$\overline{u}(z) - \overline{u}(z_0) = \frac{u_*}{k_v} \left(\ln \frac{z}{z_0} + 5 \frac{z - z_0}{L} \right)$$

[0.5] where by definition of z_0, $\overline{u}(z_0) \equiv 0$

2. Turbulence energetics

Marks could be gained as indicated...

[0.5] The kinetic energy of the turbulence is measured by the "TKE," defined $k = (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$

[0.5] Necessity for an energy supply to overcome loss of turbulent kinetic energy by viscous dissipation ϵ) to heat (internal energy)

[0.5] Cascade of energy – transfer across the spectrum from production scales to dissipation scales

[0.5] TKE produced by "shear production" (identify in given TKE eqn)

[0.5] TKE produced (unstable strat.) or removed (stable strat.) by "buoyant production" (identify in given TKE eqn) depending on the sign of the heat flux

[0.5] TKE may also be transferred from one layer or elevation to another, by way of the transport terms

[0.5] If the transport terms are negligible (as is often assumed) then "local equilibrium" prevails

[0.5] Fig. (1) shows the energy conversion pathways in a horizontally-uniform PBL:

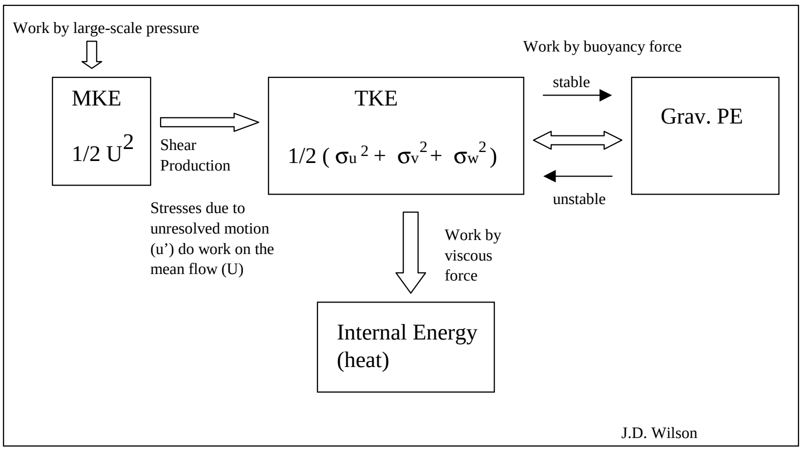

Figure 1: *Pools of Energy and transformation pathways between them*

[0.5] the TKE as measured by k does not distinguish between energy in the u', v' and w' pools, but there are budget equations for each of these separately: and in particular there appear in these equations terms of form $\overline{p' \, \partial u'_\alpha / \partial x_\alpha}$ (no summation) that are known as "redistribution terms" and which when summed across all three equations sum to zero (thus, no redistribution term in the TKE eqn). These terms shuffle TKE from component to component, and are considered to act to try to equalize the distribution of TKE across the three component pools

[0.5] this notion is captured by Fig. (2).

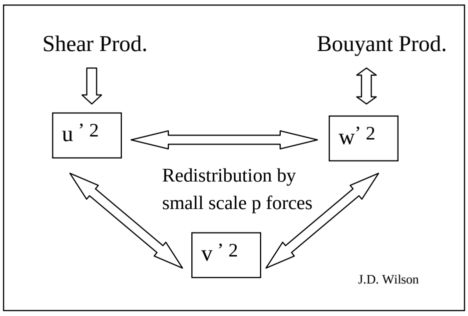

Figure 2: *TKE sources and sinks, and redistribution of TKE amongst the variance pools*

[0.5] the flux Richardson number R_i^f (given as data) is the ratio of buoyant to shear production, so that $|R_i^f| = 1$ signifies equality of contributions

[0.5] furthermore $R_i^f \equiv \frac{1}{\phi_m} \, \frac{z}{L}$ where $\phi_m \to 1$ as $R_i^f \to 0$. Therefore in moderate stability the height $z = |L|$ is the height where buoyant and shear production have roughly equal magnitude

[0.5] viscous dissipation drains energy from each of the three pools

3. Heat budget

[0.5] assume no sources are present ($Q = 0$)

[1.0] specify the sensible heat flux vector as $\vec{u}\,\theta - \kappa\,\nabla\theta$, and drop the molecular conduction term, thus

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial x_j}\left(u_j\,\theta \right)$$

[1.0] Reynolds average, noting averaging commutes with differentiation

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial}{\partial x_j}\,\overline{u_j\,\theta}$$

[1.0] discard terms in $\partial/\partial x, \partial/\partial y$ (horizontal homogeneity)... *Note: logically important to average* before *applying the simplification that flow from assuming horizontal homogeneity*

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{w}\,\overline{\theta} + \overline{w'\theta'}\right)$$

[1.0] discard term in $\overline{w}$ which vanishes as a consequence of the combined circumstances that the velocity field is incompressible and the flow is horizontally homogeneous

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial\,\overline{w'\theta'}}{\partial z}$$

[0.5] to within the constant factor $\rho_R c_p$, the surface value of $\overline{w'\theta'}$ is the sensible heat flux which is constrained by the surface energy budget