EAS572, "The Atmospheric Boundary Layer" <u>Mid-term Exam</u> 23 Oct., 2007

<u>Professor</u>: J.D. Wilson <u>Time available</u>: 60 mins <u>Value</u>: 15%

Note: Relevant equations (including the TKE equation) and data are given at the back of the exam. For all questions, please assume a local Cartesian coordinate system oriented such that x coincides with the mean wind direction (assumed height-independent) in the atmospheric surface layer (ASL), and z with the local vertical. Indices $(1, 2, 3)$ are to be interpreted as denoting respectively the (x, y, z) components, e.g. $\vec{u} \equiv u_i \equiv (u_1, u_2, u_3) \equiv (u, v, w)$, and the summation convention applies for repeated alphabetic subscripts (e.g. $u_j u_j$). Symbols z_0, δ denote the surface roughness length and the ABL depth.

Multichoice $(10 \times \frac{1}{2}\% = 5\%)$

1. Setting $i = j = 1$, the term $\partial u_i / \partial x_j$ evaluates to ____

   (a) 0
   (b) 1
   (c) $\partial u / \partial x$ ✓✓
   (d) $\nabla \cdot \vec{u}$
   (e) 3

2. The term $u_j \frac{\partial \phi}{\partial x_j}$ (where ϕ is an arbitrary scalar) expands to or may be alternately written as ____ . Assuming non-divergence of the velocity field, it may also be written ____

   (a) $u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial z}$; $\frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y} + \frac{\partial w\phi}{\partial z}$ ✓✓
   (b) $\vec{u} \cdot \nabla\phi$; $u \frac{\partial u\phi}{\partial x} + v \frac{\partial v\phi}{\partial y} + w \frac{\partial w\phi}{\partial z}$
   (c) $\vec{u}\, \nabla \cdot \phi$; $\frac{\partial u\phi}{\partial x} + \frac{\partial v\phi}{\partial y} + \frac{\partial w\phi}{\partial z}$
   (d) $\vec{u} \cdot \nabla\phi$; 0
   (e) $\vec{u}\, \nabla \cdot \phi$; $\nabla \cdot (\vec{u}\, \phi)$

3. If a neutrally-stratified wall shear layer is horizontally-homogeneous and in local equilibrium, the rates of shear production $[P_s = -\overline{u'w'}\ \partial\overline{u}/\partial z]$ and viscous dissipation $[\epsilon = \epsilon(z)]$ of turbulent kinetic energy balance. In such a layer:

   (a) $\epsilon =$ const.
   (b) $\overline{u'w'} = 0$
   (c) $P_s = \epsilon = 0$
   (d) $k_v\, z\, \epsilon(z) / u_*^3 = 0$
   (e) $k_v\, z\, \epsilon(z) / u_*^3 = 1$ ✓✓

4. A "quadrant analysis" sorts the instantaneous values of $(u',\ w')$ into four bins according to the four possible sign combinations,

$$Q_\alpha = \begin{cases} Q_1, & (u' > 0, w' > 0) \ \text{: outward interaction} \\ Q_2, & (u' < 0, w' > 0) \ \text{: ejection} \\ Q_3, & (u' < 0, w' < 0) \ \text{: inward interaction} \\ Q_4, & (u' > 0, w' < 0) \ \text{: sweep, gust} \end{cases}$$

In a turbulent boundary layer, the dominant quadrants are

(a) Q_1, Q_2

(b) Q_1, Q_3

(c) Q_2, Q_3

(d) Q_2, Q_4 ✓✓

(e) Q_1, Q_4

5. In the ASL the covariance $\overline{u'w'} \equiv -\ u_*^2 < 0$, and during daytime summer normally $\overline{w'T'} > 0$. The statistic $\overline{u'T'}$ is the ____ and under the given conditions its sign is likely to be ____ . (Hint: consider the likely signs of u', w', T' assuming appropriate qualitative profiles of $\overline{u}, \overline{T}$)

(a) streamwise eddy heat flux density; positive

(b) streamwise eddy heat flux density; negative ✓✓

(c) streamwise eddy heat flux density; same as that of the flux $\overline{u}\ \overline{T}$ carried by the mean flow

(d) TKE dissipation rate; positive

(e) rate of shear production of TKE; positive

6. Reynolds averaging the property f^3 gives

(a) $\overline{f^3} = \overline{f'}^{\,3}$

(b) $\overline{f^3} = \overline{f}^{\,3}$

(c) $\overline{f^3} = 0$

(d) $\overline{f^3} = \overline{f}^{\,3} + \overline{f'^3}$

(e) $\overline{f^3} = \overline{f}^{\,3} + 3\,\overline{f}\ \overline{f'^2} + \overline{f'^3}$ ✓✓

7. Remembering the x-axis is aligned with the mean wind, the components of which are therefore $\overline{u}_i \equiv (\overline{u}, 0, 0)$, "shear production" feeds kinetic energy directly into

   (a) $\overline{u}$

   (b) $\overline{u'w'}$

   (c) σ_w^2

   (d) σ_u^2 ✓✓

   (e) σ_u^2 and σ_v^2

8. In his 1915 paper on the potential temperature profile in the atmospheric boundary layer (ABL), G.I. Taylor developed the equation (here given in slightly revised notation)

$$\frac{\partial \overline{\theta}}{\partial t} = \mathcal{W}\mathcal{D}\, \frac{\partial^2\, \overline{\theta}}{\partial z^2}$$

   where $\mathcal{W}\mathcal{D}$ constitutes an effective eddy thermal diffusivity (a specific definition is given by Taylor). The right hand side should best be interpreted as being ____

   (a) the radiative flux divergence

   (b) the divergence of the vertical heat flux carried by molecular conduction

   (c) the divergence of the vertical convective heat flux carried by eddies ✓✓

   (d) latent heating/cooling associated with change of phase of water

   (e) $\epsilon/(\rho_0 c_p)$, where ϵ is the rate of conversion of turbulent kinetic energy to heat

9. Regarding Taylor's treatment, if (from today's perspective) we chose to describe the ABL using the 'diffusion' formulation (which remains the case with most of the present day numerical weather models), then an appropriate generalization would be

   (a) identify $\mathcal{W}\mathcal{D}$ with the MOST eddy diffusivity $k_v u_* z\, \phi_h^{-1}$

   (b) treat the eddy diffusivity as height dependent, replacing the right hand side with $\frac{\partial}{\partial z}\left[K(z)\,\frac{\partial \overline{\theta}}{\partial z}\right]$ ✓✓

   (c) identify $\mathcal{W}$ with the standard deviation of vertical velocity (σ_w) and $\mathcal{D}$ with boundary-layer depth (δ)

   (d) identify $\mathcal{W}$ with the mean vertical velocity ($\overline{w}$) and $\mathcal{D}$ with the lengthscale $\overline{w}/f$ (where f is the Coriolis parameter with units s^{-1})

   (e) identify $\mathcal{W}$ with the Geostrophic windspeed ($|\vec{U}_g|$) and $\mathcal{D}$ with the lengthscale $|\vec{U}_g|/f$

10. The Obukhov length L (defined in the Data section) may also be written ____

(a) $\frac{u_*^3\, T_0}{k_v\; g\; \theta_*}$

(b) $\frac{u_*^3\, \theta_*}{k_v\; g\; T_0}$

(c) $k_v z$

(d) $\frac{u_*^2\, T_0}{k_v\; g\; \theta_*}$ ✓✓

(e) $k_v \sqrt{\delta\; z_0}$

1 Short answer $(2\,\text{x}\,5\% = 10\%)$

Instructions: Please answer any **two** of the following three questions. Organization of your answers is important to ensure clarity. Use diagrams wherever they may be helpful and especially if they demonstrate your interpretation of the situations/questions as posed here verbally. Be sure to state any assumptions or simplification you make. ***No page limit applies.***

1. Starting from the conservation equation

$$\frac{\partial \phi}{\partial t} = -\; \nabla \cdot \vec{F} + Q$$

and making the identification $\phi = \rho_v$ (the absolute humidity), derive a conservation equation for the *mean* humidity $\overline{\rho}_v$ in the horizontally-homogeneous ABL. Please note (and justify) all simplifications, restrictions and assumptions you make.

Skeleton response: Points could be accumulated to a total of 5 as follows:

- identify the flux $\vec{F} = \vec{u}\; \rho_v \;-\; \mathcal{D}\, \nabla \rho_v$ [**1**]
- correctly apply Reynolds averaging,

$$\overline{\frac{\partial\, u_j\, \rho_v}{\partial x_j}} \equiv \frac{\partial\, \overline{u_j\, \rho_v}}{\partial x_j} \equiv \frac{\partial}{\partial x_j}\; \left(\overline{u}_j\; \overline{\rho}_v +\; \overline{u'_j\, \rho'_v}\right) \quad [\mathbf{2}]$$

- correctly carry out the tensor summation and invoke the assumption of horizontal homogeneity[1] (i.e. neglect horizontal derivatives of *averages* and set $\overline{w} = 0$) to obtain final result:

$$\frac{\partial\, \overline{\rho}_v}{\partial t} \;=\; -\; \frac{\partial\, \overline{w' \rho'_v}}{\partial z} \quad [\mathbf{2}]$$

These steps, correctly carried out without any serious lapse of notation, would gain full marks (with no penalty if the molecular transport term and/or the source term were retained). Any loss of marks in carrying out the derivation could be compensated by restoration of one mark for noting one or both of these steps:

[1] Several students did not recognize that setting $\overline{w} = 0$ is a legitimate step if one has assumed horizontal homogeneity.

- neglect molecular diffusion, with justification that except adjacent to the surface the convective flux is overwhelmingly larger
- neglect source term, with justification that you are restricting to case where there are no phase changes
- explicitly noting that, as a step in the derivation, one may use (any one of) the relations

$$\begin{aligned} \partial u_j/\partial x_j &= 0 \\ \partial \overline{u}_j/\partial x_j &= 0 \\ \partial u'_j/\partial x_j &= 0 \end{aligned}$$

2. Draw schematic vertical profiles of the following dimensionless statistics within the horizontally-homogeneous atmospheric surface layer over a bare soil surface:

$$\frac{\overline{u}}{u_*},\quad \frac{\sigma_u}{u_*},\quad \frac{\sigma_w}{u_*},\quad \frac{\overline{u'w'}}{u_*^2},\quad \frac{\overline{w'T'}}{u_* T_*},\quad \frac{k_v z\,\epsilon}{u_*^3}$$

Since these properties are all dimensionless, you may (if you wish) plot more than one variable on a given graph. Your height axis/axes (the "ordinate" of your graph) may be linear or logarithmic, according to your choice: and should cover the range $z/|L| \leq 2$. Your abscissa/abscissae should indicate approximate numerical values. Aim to convey the qualitative height trends.

Skeleton response: Please see Fig.(1) at back for instructor's sketches. In hindsight, this was a time-consuming and difficult question; an unintended complication was the implicit invitation (or strictly, necessity) to deal with both possibilities for the sign of L. A common problem in the responses was the absence of a scale on the abscissa, which absence for purposes of marking had to be taken to indicate the student did *not know* the magnitude and/or range of the variable in question. Don't overlook to mark in numerical value at obvious special points on the axis, e.g. $\overline{u}/u_* = 0$ at $z = z_0$.

Question was marked by assigning $\frac{1}{2}$% for every correct point, e.g. $\frac{1}{2}$ for noting wind-speed increases with height, another $\frac{1}{2}$ if the shear was shown as stronger at lower heights, another $\frac{1}{2}$ for a correct scale marking, (etc).

- $\overline{u}/u_*$: Varies linearly with $\ln(z/|L|)$ for small $z/|L|$, irrespective of the sign of L. Curvature of the profile for finite $z/|L|$ depends on the sign of L through $\phi_m(z/L)$, viz.

$$\frac{\partial \overline{u}}{\partial \ln z} = \frac{u_*}{k_v}\,\phi_m\left(\frac{z}{L}\right) \tag{1}$$

  with stronger shear on the stable side (for which $\phi_m \approx 1 + 5z/L$, Lec. 8). Finding a scale for the magnitude of $\overline{u}/u_*$ hinges on remembering that typical near-ground windspeeds are a few $\mathrm{m\,s^{-1}}$ while corresponding friction velocities are an order of magnitude smaller.

- σ_w/u_* : Order unity for small $z/|L|$ (Lec. 8). Increases with increasing $z/|L|$ in the unstable case (Lec. 8)

- σ_u/u_* : Also order unity for small $z/|L|$ (Lec. 8), but larger than σ_w/u_* (Lec. 8, Lec. 9). Increases with increasing $z/|L|$ in the unstable case (Lec. 8). Fractional rate of change with height in the ASL is smaller than that of σ_w/u_*, owing to the fact that the u spectrum contains energy at low frequency that is missing from the w spectrum
- $\overline{u'w'}/u_*^2$: Order -1 by definition of u_*. Exactly -1 for all height $z/|L| \le 2$ by definition, if one takes *literally* the notion of the surface layer being a constant stress layer. In truth, its magnitude decreases with increasing $z/|L|$, however the change across the surface layer is not more than about 10-20 %
- $\overline{w'T'}/(u_* T_*)$: Order -1 by definition of u_* and T_* (recall $T_* = -\overline{w'T'}/u_*$, given as data). Exactly -1 for all height $z/|L| \le 2$ by definition, if one takes *literally* the notion of the surface layer being a constant flux layer. In truth, its magnitude decreases with increasing $z/|L|$, balancing the temperature trend; however the change across the surface layer is not more than about 10-20 %
- $\frac{k_v\, z\, \epsilon}{u_*^3}$: Order 1. Exactly unity in a neutral surface layer ($|L| = \infty$), if one assumes the TKE budget is in local equilibrium (see given data). Increases (decreases) with $z/|L|$ on the unstable (stable) side, balancing the sum of positive (negative) buoyant production and shear production

3. The mean heat budget for an cloudless, unsaturated, horizontally-homogeneous ABL may be written

$$\frac{\partial \overline{\theta}}{\partial t} = -\frac{\partial \overline{w'\,\theta'}}{\partial z}$$

Assuming a well-mixed, very convective ABL (mean temperature $\overline{\theta}_m$), and neglecting any entrainment heat flux, compute the mean surface kinematic heat flux density over the interval if the mean temperatures at 11:00 and 13:00 Local Standard Time were respectively $\overline{\theta}_m = 18$ K and $\overline{\theta}_m = 23$ K. Comment on the magnitude of this heat flux in the context of its being a term in the surface energy balance, assuming $\rho_0\, c_p \approx 10^3\ \mathrm{J\,m^{-3}\,K^{-1}}$.

Skeleton response: This could not be done without knowing[2] (or assuming a value for) the boundary-layer depth, δ

- Integrate w.r.t. z to get the integral balance

$$\frac{\partial}{\partial t}\int_0^{\delta} \overline{\theta}\, dz = (\overline{w'\,\theta'})_0$$

(here I used the given fact that the entrainment heat flux $(\overline{w'\,\theta'})_\delta = 0$)

- Because we are given that this ABL is well-mixed, we make the approximation

$$\int_0^{\delta} \overline{\theta}\, dz = \delta\, \overline{\theta}_m \qquad (2)$$

[2] Students were invited at the start of the exam to ask the instructor for assistance, in case any question was unclear or ambiguous.

and this leads to

$$\frac{\partial}{\partial t}\,\delta\,\overline{\theta}_m = \delta\,\frac{\partial\,\overline{\theta}_m}{\partial t} + \overline{\theta}_m\,\frac{\partial\,\delta}{\partial t} = (\overline{w'\,\theta'})_0$$

- To keep life simple, let's neglect the term in $\partial\delta/\partial t$, so that

$$\delta\,\frac{\partial\,\overline{\theta}_m}{\partial t} = \delta\,\frac{5}{7200} \approx (\overline{w'\,\theta'})_0$$

- With (e.g.) $\delta = 10^3$ m, we have $(\overline{w'\,\theta'})_0 = 5000/7200 = 0.7\ \mathrm{K\,m\,s^{-1}}$ or $Q_{H0} \sim 700\ \mathrm{W\,m^{-2}}$
- This is an unreasonably large figure for Q_{H0}, since even if the surface were dry this would demand a net radiation of the same order.
- Including the term in $\partial\delta/\partial t$ (or taking a deeper estimate for δ) must lead to an even larger figure
- Apparently the stated rate of warming is incompatible with other assumptions, e.g. absence of advection (advection cannot be accounted for in a 1-D balance) or the rather deep ABL.

Data

Note: ρ_0, T_0 are the reference density and reference (Kelvin) temperature. For the purposes of the exam, you may consider the quantities $\overline{w'T'}$ and $\overline{w'\theta'}$ identical (the kinematic heat flux density).

- The turbulent kinetic energy (TKE, "k") equation, assuming steady state and horizontal uniformity, and assuming a uni-directional mean flow $(\overline{u}, 0, 0)$, is:

$$\frac{\partial k}{\partial t} = 0 = -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} + \frac{g}{T_0}\,\overline{w'T'} - \frac{\partial}{\partial z}\,\overline{w'\ (p'/\rho_0 + \ e'\ )}\ -\ \epsilon$$

where $e' \equiv \left(u'^2 + v'^2 + w'^2\right)/2$ and ϵ is the TKE dissipation rate.

- Obukhov length $L = -\,u_*^3\,T_0\,\left(k_v\,g\,\overline{w'T'}\right)^{-1}$
- The height-gradients in mean windspeed and mean potential temperature, according to the Monin-Obukhov similarity theory (MOST), are

$$\begin{aligned}\frac{k_{vm} z}{u_*}\,\frac{\partial \overline{u}}{\partial z} &= \phi_m\left(\frac{z}{L}\right)\\ \frac{k_{vh} z}{\theta_*}\,\frac{\partial \overline{\theta}}{\partial z} &= \phi_h\left(\frac{z}{L}\right)\end{aligned}$$

where the temperature scale is $\theta_* \equiv -\,\overline{w'\theta'}/u_*$. It is acceptable to assume equality of the von Karman constants ($k_{vm} = k_{vh} = k_v = 0.4$), and that in the neutral limit, defined by $z/L \to 0$, the universal functions evaluate as $\phi_m(0) = \phi_h(0) = 1$.

- The ‘surface energy balance’ on a reference plane at the base of the atmosphere is expressed by the equation
$$Q^* \ = \ Q_H \ + Q_E \ + Q_G$$
where all fluxes are in [W m^{-2}]. Sign convention: Q^* the net radiation, positive if directed towards the surface; Q_H, Q_E the sensible and latent heat fluxes, positive if directed from the surface towards the atmosphere; Q_G the ‘soil’ heat flux, positive if directed from the surface into ground/lake/ocean. The Bowen ratio $\beta = Q_H/Q_E$ quantifies the partitioning of the energy surplus (or deficit) $Q^* - Q_G$ by the surface.

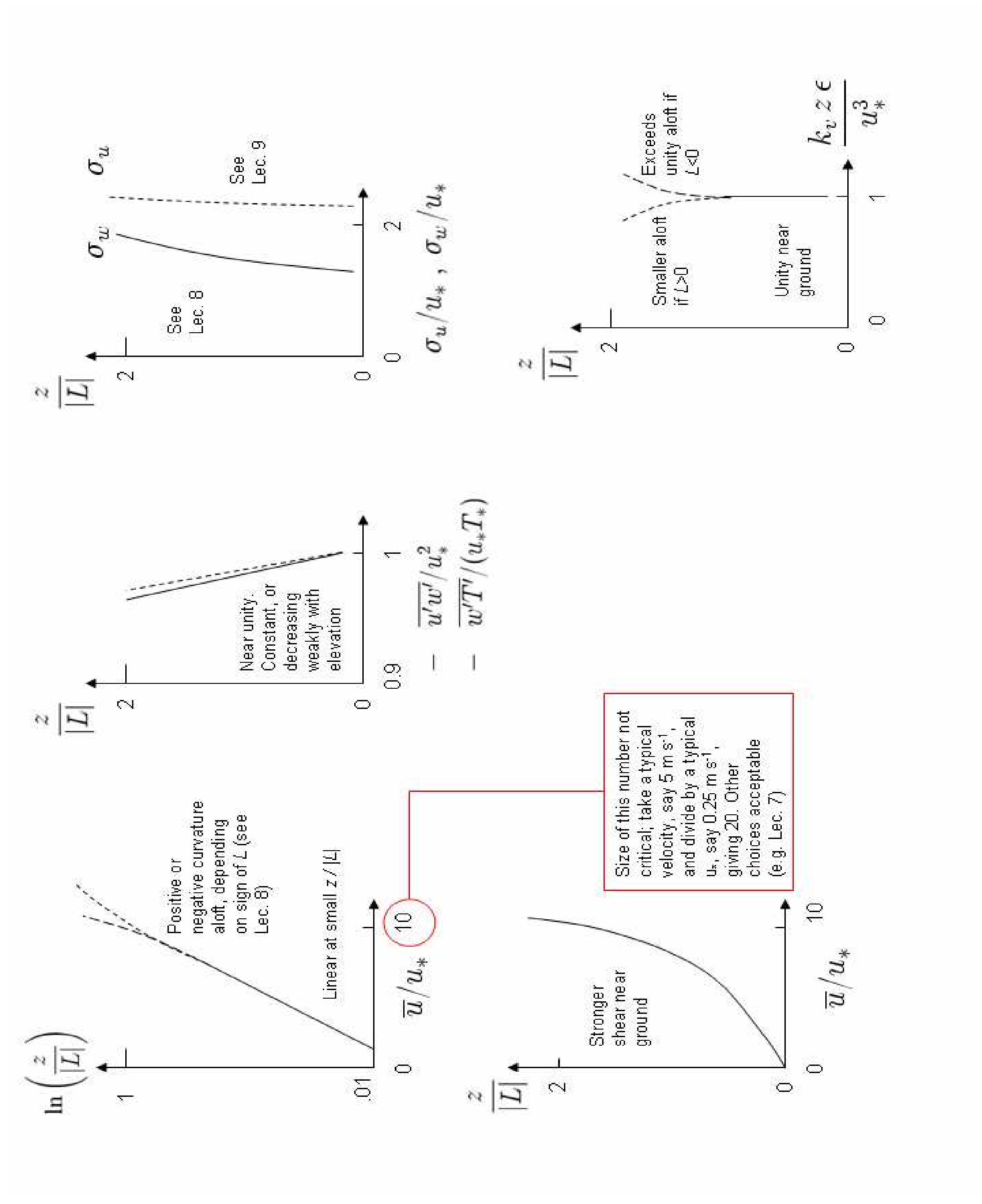

Figure 1: *Schematic profiles of ASL statistics.*