COMMENTS ON A RELATIONSHIP BETWEEN FLUID AND IMMERSED-PARTICLE VELOCITY FLUCTUATIONS PROPOSED BY WALKLATE (1987)

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Walklate (1987) has proposed that the ratio σ_p^2/σ^2 of heavy particle velocity variance to the driving fluid turbulent velocity variance is given by

$$\sigma_p^2/\sigma^2 = 1/(1 + c\tau_p/\tau_L)\,, \tag{1}$$

where τ_p is the particle Stokesian response time, τ_L is the fluid Lagrangian time-scale, and c is an empirical constant. We suggest that this equation is not accurate. We will give some background discussion, followed by a dimensional analysis indicating the possibility that this equation for σ_p^2/σ^2 is an over-simplification. Finally, our criticism of the expression will be strengthened by an examination of experimental results.

Heavy particles moving in a turbulent flow exhibit velocity statistics which differ from the corresponding (Lagrangian) fluid element velocity statistics because of:

(a) Particle inertia, which reduces particle response to high-frequency forcing by the surrounding fluid in a manner which is well understood for the case where the driving velocity is position-independent and varies only in time (the infinite time-varying eddy case, or 'oscillating Stokes flow').

(b) Particle bias motion due to the presence of a gravitational (or other body) force.

Both particle inertia and particle response to a body force can result in a 'crossing trajectories effect' – the particle cuts across regions of correlated fluid motion (eddies).

There are two possible Lagrangian approaches to the calculation of heavy particle dispersion. The first option is to calculate both the particle velocity u_{pi} and the fluid velocity u_i in the immediate environment of the particle, these velocities being linked by the equation of motion for the particle. The core of the problem then lies in the determination of the time series of the forcing velocity u_i, which is neither an Eulerian nor a fluid Lagrangian series since both the position of interest and the fluid element occupying that position are changing. An example of this approach is the work of Hunt and Nalpanis (1985).

The second Lagrangian approach is to disregard the fluid: then if the particle velocity statistics are known, one may either apply Taylor's (1921) analytical result (for the case of a homogeneous motion) or, in the case of inhomogeneous turbulence, carry out a trajectory simulation. The recent papers by Walklate (1986, 1987) belong to this second Lagrangian category.

Provided that the particle density ρ_p is very much larger than the fluid density ρ, the equation of motion for a spherical particle may be written (Schlichting, 1968)

$$\frac{\mathrm{d}u_{pi}}{\mathrm{d}t} = \frac{u_i - u_{pi}}{\tau_p}\,(1 + \tfrac{3}{16}R_e) - g_i\,. \tag{2}$$

Here R_e is a time-dependent slip Reynolds number $R_e = |\mathbf{u}_p - \mathbf{u}|\, d/\nu$, and the above equation is valid for $R_e \lesssim 5$. The particle time constant or 'Stokesian response time' is

$$\tau_p = \frac{\rho_p d^2}{18\mu}\,,$$

where d is the particle diameter and μ is the dynamic viscosity.

For a given sinusoidal variation of u_i in time alone, Equation (2) may be solved for the oscillation in u_{pi} provided the Reynolds-number dependent correction is dropped (i.e., for low R_e) – this is called the 'oscillating Stokes flow problem'. The resulting complex frequency response function is (for simplicity of notation reducing to a one-dimensional motion and dropping the gravitational acceleration)

$$W(f) = F\{u_p\}/F\{u\} = 1/(1 + j2\pi f\tau_p)\,, \tag{3}$$

where $j = \sqrt{-1}$, $F\{\ \}$ denotes the Fourier transform*, and f is the frequency of the oscillation in u. Then the 'power gain', i.e., the ratio (amplitude^2 of particle velocity/amplitude^2 of air velocity) is

$$G(f) = |W(f)|^2 = \frac{1}{1 + (2\pi f\tau_p)^2}\,. \tag{4}$$

If the particle is now considered to be driven by a spectrum of fluid motion $S(f)$, where the fluid velocity variance $\sigma^2 = \overline{u^2} = \int_0^\infty S(f)\,\mathrm{d}f$, the particle velocity spectrum $S_p(f)$ is related to the fluid velocity spectrum by

$$S_p(f) = G'(f)S(f)\,, \tag{5}$$

where the power gain function $G'(f)$ has been given a prime to emphasize the restricted validity of Equation (4) which followed from the adoption of a spatially-invariant fluid velocity field and a small slip Reynolds number. In general, the fluid velocity spectrum will imply a spatially-varying velocity pattern.

The particle velocity variance is

$$\sigma_p^2 = \overline{u_p^2} = \int_0^\infty S_p(f)\,\mathrm{d}f\,.$$

* E.g., a function $a(t)$ has Fourier transform $F\{a\} = \int_{-\infty}^{\infty} a(t)e^{-j2\pi ft}\,\mathrm{d}t$.

Here and elsewhere, the overbar is used to denote an average value. More explicit results depend on adoption of a form for $S(f)$. Hinze (1975) shows that if $S(f)$ is the spectrum corresponding to an exponential Lagrangian autocorrelation function for the fluid velocity

$$R(\xi) = \overline{u(t)u(t+\xi)}/\overline{u^2} = e^{-|\xi|/\tau_L}$$

and if $G'(f)$ is assumed to be given by Equation (4) in spite of the fact that the fluid velocity field corresponding to this spectrum may not be spatially invariant, then

$$\sigma_p^2/\sigma^2 = 1/(1 + \tau_p/\tau_L)\,. \tag{6}$$

For future reference, let us note that the spectrum and the autocovariance function $C(\xi) = \overline{u^2}R(\xi)$ are a Fourier transform pair, so Equation (5) leads to a particle autocovariance function

$$C_p = F^{-1}\{G'(f)\} * C \tag{7}$$

where the * denotes the convolution interval. In the present case, for $G'(f)$ given by Equation (4), the inverse transform of the power gain is

$$g(\xi) = F^{-1}\{G(f)\} = \frac{1}{2\tau_p}\exp(-|\xi|/\tau_p)\,. \tag{8}$$

Meek and Jones (1973) gave a statistical analysis of heavy particle motion in homogeneous turbulence (as distinct from a system in which the fluid velocity is spatially invariant) and deduced that the ratio of particle velocity variance σ_p^2 to fluid velocity variance σ^2 is given by Equation (6), i.e., is controlled only by τ_p/τ_L. Walklate (1986) obtained an incorrect result when combining the assumptions of an exponential fluid autocorrelation function and the simple power gain function (Equation (4)) appropriate to oscillating Stokes' flow – a factor of 2π appears multiplying τ_p/τ_L in Equation (6) (his Equation (14)). His analysis appears to go awry when an incorrect result is obtained for the inverse Fourier transform of the power gain function (his Equation (11)), the correct result being our Equation (8). In consequence, his deduced particle autocorrelation function and particle velocity variance (his Equations (12) and (14)) are wrong – the authors, in agreement with Hinze (1975), obtain for the particle autocorrelation

$$R_p(\xi) = \frac{\tau_p e^{-\xi/\tau_p} - \tau_L e^{-\xi/\tau_L}}{\tau_p - \tau_L}$$

and for the particle velocity variance, Equation (6).

In his later paper, Walklate (1987) introduces a coefficient c into his particle power gain function (his Equation (4))

$$G(f) = \frac{S_p(f)}{S(f)} = \frac{1}{1 + (c2\pi f\tau_p)^2}\,,$$

'to extend ... [the oscillating Stokes' flow solution having $c = 1$] ... beyond the Stokes' law region'. His result for the particle velocity variance is then Equation (1).

Lagrangian simulations of particle dispersion were described, and the dependence of the predictions on the value prescribed for the constant c in Equation (1) was shown. For neither of the values chosen ($c = 2$, $c = 4$) was the agreement with the diverse observations (for a range of particle types) very good.

Features of Walklate's simulations other than Equation (1) may have caused or compounded the discrepancies relative to observations. However, we believe that σ_p^2/σ^2 is far from being a function of τ_p/τ_L alone, and that 'c' is an ineffective cure-all because it must account for other factors in addition to the possibility that the slip Reynolds number is not small and constant in time – most importantly the fact that the fluid velocity varies in space (the crossing trajectories effect).

A dimensional analysis is helpful. A comprehensive group of governing variables for heavy particle motion in homogeneous isotropic turbulence should include:

(a) Variables of the fluid; density ρ, kinematic viscosity ν.

(b) Variables of the turbulence; velocity* standard deviation σ, rate of dissipation of turbulent kinetic energy ε (from which may be formed (large) length and time-scales $L = \sigma^3/\varepsilon$, $T = \sigma^2/\varepsilon$).

(c) Variables of the particle; density ρ_p, diameter d, (and particle characteristic velocity σ_p, which we would like to predict).

(d) External factors; acceleration due to gravity, g.

Note that the above choice includes by rearrangement the (small) Kolmogorov scales of motion $v_K = (\nu\varepsilon)^{1/4}$, $\eta_K = (\nu^3/\varepsilon)^{1/4}$, and $t_K = \eta_K/v_K$. If the above 8 variables (having 3 dimensions) do indeed govern the problem, we may expect to find a relationship between 5 non-dimensional ratios (Bridgman, 1922),

$$\sigma_p/\sigma = F\left(\rho_p/\rho,\ \sigma/v_K,\ \frac{\sigma d}{\nu},\ \frac{g t_K}{\sigma}\right).$$

Alternative expressions based on the same 8 variables must be equivalent. Other ratios which may spring to mind as being relevant must be accounted for by recombination of the given ratios, e.g.,

$$d/\eta_K = \frac{(\sigma d/\nu)}{\sigma/v_K},$$

and, for $\rho_p \gg \rho$ (so that $\tau_p = d^2\rho_p/18\rho\nu$)

$$\frac{\tau_p}{T} \propto \left(\frac{\rho_p}{\rho}\right)\left(\frac{\sigma d}{\nu}\right)^2 \Big/ \left(\frac{\sigma}{v_K}\right)^4.$$

* Since the driven particle moves in space and relative to the driving fluid element, the driving fluid velocity is neither Eulerian nor Lagrangian.

This last ratio (τ_p/T) is equivalent to the τ_p/τ_L of other analyses. The ratio σ/v_K pertains to the eddy size range. Though it is not explicitly shown, a Reynolds number based on the magnitude of the relative velocity between particle and fluid and particle size must be considered to be involved through the ratios chosen; this is the R_e appearing in Equation (2) and pertaining to the nature of the fluid drag on the particle.

Now that a wider range of possible influences upon σ_p/σ has been suggested, it remains to seek experimental support for or against the hypothesis that σ_p/σ depends upon τ_p/τ_L alone.

Snyder and Lumley (1971; hereafter SL) and Wells and Stock (1983) have presented observations of heavy particle dispersion in decaying turbulence downstream of a wind tunnel grid. Comprehensive turbulence data are given, in addition to the particle velocity and dispersion data. In many respects, the two experiments were very similar, though Wells and Stock used laser anemometry to deduce velocities rather than photographs, and were able to control the effective gravitational acceleration by using charged particles in an electric field.

In each case, theory suggests that the turbulence time-scales should increase linearly with distance from the grid. Reported values of the Kolmogoroff time-scale conformed closely to this expectation. Also in accordance with expectation, the fluid turbulent velocity scale σ was observed to vary with $(x - x_0)^{-1/2}$, while the Taylor micro (length) scale λ varied with $(x - x_0)^{1/2}$ so that $\sigma\lambda \sim$ constant (here x_0 is a virtual origin). There seems little reason to doubt SL's suggestion that the Lagrangian time-scale τ_L varied linearly with $x - x_0$, so that from their Figure 14, and allowing for this linear variation, we may infer that for their experiment

$$\tau_L \cong 0.1(x/M - 14)/(73 - 14).$$

(Here x/M is streamwise distance normalized by the grid mesh length $M = 2.54$ cm (1 inch), and $x/M = 14$ is our choice for the virtual origin of the decay.)

Particle cross-stream velocity variances at different positions in the flow (and therefore in environments with different τ_L) may be obtained from SL Figure 10 and corresponding Eulerian cross-stream velocity variances are given by SL Equation (10). Table I gives the values deduced for two streamwise locations, 0.1 and 0.3 s from station 1 (which lay at $x/M = 68$). The time needed for a particle to travel between these two positions is only of the order of 1 τ_L (or 4 to 10 τ_p, depending on the particular particle) which is not as large as one might wish. However, one may reasonably conclude:

(i) The ratios σ_p^2/σ^2 rank inversely with τ_p.

(ii) For each particle, σ_p^2/σ^2 is the same at both positions, while τ_p/τ_L differs from one position to the other by a sizeable percentage.

Hence, the dependence of σ_p^2/σ^2 on τ_p/τ_L alone suggested by Meek and Jones and Walklate is not supported by the SL experiment. The writers have been unable to isolate a single dimensionless ratio against which σ_p^2/σ^2 correlates well (pooling the SL data with the WS data for particles charged so as to produce zero external force). In fact, there is a reasonable relationship between σ_p^2/σ^2 and the dimensional ratio τ_p/σ (or

TABLE I

Ratio of particle-to-fluid velocity variances and corresponding ratio of particle time-scale to turbulence time-scale for two locations and three particle types: deduced from data of Snyder and Lumley (1971), x/M being their dimensionless streamwise location.

	Location					
	0.1 s from station 1 $x/M \approx 94$ $\tau_L \approx 0.13$ s			0.3 s from station 1 $x/M \approx 145$ $\tau_L \approx 0.22$ s		
	σ_p^2/σ^2	τ_p/τ_L	'Required' c	σ_p^2/σ^2	τ_p/τ_L	'Required' c
Corn pollen $\tau_p = 0.020$ s	0.55	0.15	5.5	0.53	0.09	9.9
Solid glass $\tau_p = 0.045$ s	0.33	0.35	5.8	0.33	0.20	10.2
Copper $\tau_p = 0.049$ s	0.30	0.38	6.1	0.27	0.22	12.3

equally τ_p/v_K) but no normalizing acceleration having streamwise invariance could be found (apart from g, whose use would be absurd, allowing no inertia effect in the absence of a gravitational field).

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