Numerical Modelling of the Turbulent Flow Developing Within and Over a 3-D Building Array, Part II: A Mathematical Foundation for a Distributed Drag Force Approach

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Abstract. In this paper, we lay the foundations of a systematic mathematical formulation for the governing equations for flow through an urban canopy (e.g., coarse-scaled building array) where the effects of the unresolved obstacles on the flow are represented through a distributed mean-momentum sink which, in turn, implies additional corresponding terms in the transport equations for the turbulence quantities. More specifically, a modified k-ϵ model is derived for the simulation of the mean wind speed and turbulence for a neutrally-stratified flow through and over a building array, where groups of buildings in the array are aggregated and treated as a porous medium. This model is based on time averaging the spatially-averaged Navier-Stokes equation, in which the effects of the obstacle-atmosphere interaction are included through the introduction of a volumetric momentum sink (representing drag on the unresolved buildings in the array).

The k-ϵ turbulence closure model requires two additional prognostic equations, namely one for the time-averaged resolved-scale kinetic energy of turbulence, $\overline{\kappa}$, and another for the dissipation rate, ϵ, of $\overline{\kappa}$. The transport equation for $\overline{\kappa}$ is derived directly from the transport equation for the spatially-averaged velocity and explicitly includes additional sources and sinks that arise from time averaging the product of the spatially-averaged velocity fluctuations and the distributed drag force fluctuations. We show how these additional source/sink terms in the transport equation for $\overline{\kappa}$ can be obtained in a self-consistent manner from a parameterization of the sink term in the spatially-averaged momentum equation. Towards this objective, the time-averaged product of the spatially-averaged velocity fluctuations and the distributed drag force fluctuations can be approximated systematically using a Taylor series expansion. A high-order approximation is derived to represent this source/sink term in the transport equation for $\overline{\kappa}$. The dissipation rate (ϵ-) equation is simply obtained as a dimensionally consistent analog of the $\overline{\kappa}$-equation. The relationship between the proposed mathematical formulation of the equations for turbulent flow within an urban canopy (where the latter is treated as a porous medium) and an

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earlier heuristic two-band spectral decomposition for parameterizing turbulence in a plant canopy is explored in detail.

Keywords: Urban winds, Disturbed winds, Wind models, Drag coefficient, Turbulence closure, Canopy flows

1. Introduction

The turbulent flow within and over urban areas covered with agglomerations of discrete buildings, often with irregular geometry and spacing, is generally very complex and possesses a fully three-dimensional statistical structure. Although the application of computational fluid dynamics (CFD) to the prediction of the mean flow and turbulence near and around a single building or within and over a regular array (or, canopy) of buildings is progressing (e.g., Smith et al., 2000; DeCroix et al., 2000; Lien et al., 2003; Lien and Yee, 2003a), this method requires extensive computational resources. Nevertheless, CFD simulations which involve the solution of the conservation equations for mass, momentum, and energy allow the prognosis of a number of velocity statistics (e.g., mean velocity, normal stresses, shear stresses, etc.) in an urban canopy. The knowledge of the structure of the mean flow and turbulence describing the complex flow patterns within and over clusters of buildings is also essential for improving urban dispersion models.

Unfortunately, the computational demands of CFD where all buildings are resolved explicitly in the sense that boundary conditions are imposed at all surfaces (e.g., walls, roofs) are so prohibitive as to preclude their use for emergency response situations which require the ability to generate an urban flow and dispersion prediction in a time frame that will permit protective actions to be taken. In view of this, we argue that for many practical applications it is convenient to consider the prediction of statistics of the mean wind and turbulence in an urban canopy that are obtained by averaging horizontally the mean wind and turbulence statistics over an area that is larger than the spacings between the individual roughness elements comprising the urban canopy, but less than the length scale over which the roughness element density changes.

This is the second in a series of three papers describing the numerical modelling of the spatially developing flow within and over a 3-D building array. In the first paper (Lien and Yee, 2003a; henceforth I), we used the Reynolds-averaged Navier-Stokes (RANS) equations in conjunction with a two-equation turbulence model (i.e., k-ϵ model) to predict the complex three-dimensional disturbed flow within and over a

3-D building array under neutral stability conditions. The simulations of the mean flow field and turbulence kinetic energy were validated with data obtained from a comprehensive wind tunnel experiment conducted by Brown et al. (2001). Here, it was demonstrated that the mean flow and turbulence kinetic energy from the numerical and physical simulations exhibited striking resemblances. In addition, the importance of the kinematic 'dispersive stresses' relative to the spatially-averaged kinematic Reynolds stresses for a spatially developing flow within and over an urban-like roughness array has been quantified using the high-resolution CFD results obtained with the high-Reynolds-number k-ϵ model. In the third paper (Lien and Yee, 2003b; henceforth III), we apply the modified k-ϵ model derived in the present paper whereby the unresolved obstacles in the 3-D building array are represented as a distributed momentum sink for the prediction of the spatially-averaged mean flow and turbulence field within and over the 3-D building array studied in I. In addition in III, we compare these predictions with those obtained by spatially-averaging the mean flow and turbulence quantities obtained from the high-resolution RANS simulation in I.

In this paper, we focus on the mathematical formulation of a numerical model for the prediction of flows within and over a building array based on an aggregation of groups of buildings in the array into a number of 'drag units', with the ensemble of units being treated as a continuous porous medium. This approach will obviate the need to impose boundary conditions along the surfaces of all buildings (and other obstacles) in the array. Wilson and Yee (2000) applied something like this approach to simulate the mean wind and turbulence energy fields in a single unit cell of the wind tunnel "Tombstone Canopy" (Raupach et al., 1986), a regular diamond staggered array of bluff (impermeable) aluminum plates, with a disappointing outcome (subsequent work showed that invoking a Reynolds stress closure did not help). We now know this may owe to the existence of (previously unsuspected) large eddies generated by the strong shear layer near the top of the canopy,[1] eddies that span more than one unit cell in the streamwise direction, and imply that imposition of an artificial condition of periodicity at the boundaries of a single cell amounts to solving a different flow problem. Belcher et al. (2003) applied a similar approach to investigate the adjustment of the mean velocity to a canopy of roughness elements using a linearized flow model (obtained by determining analytically small

[1] These large-scale (coherent) eddy structures generated at or near the canopy top have been observed using highly resolved, two-dimensional laser-induced fluorescence measurements of the fine structure of the fully space- and time-varying conserved scalar field resulting from a point-source release of a tracer within the "Tombstone Reloaded Canopy" in a water channel simulation (Yee et al., 2001).

perturbations to the undisturbed upstream logarithmic mean velocity profile induced by the drag due to an obstacle array).

There is precedent for treating drag on unresolved buildings in an urban canopy by means of a distributed momentum sink for the representation of the effects on the mean flow and turbulence arising from the form and viscous drag on canopy elements. As motivation, we recall that a similar approach has been applied over the past 50 years to the modelling of flows in plant canopies and about porous windbreaks. Although a sink or drag term has been added in an *ad hoc* fashion to the free-air mean momentum equation to model the canopy mean wind profile over a number of years (e.g., Inouye, 1963; Uchijima and Wright, 1964; Cowan, 1968; among others), it was not until 1977 that Wilson and Shaw (1977) showed how to apply a rigorous spatial-averaging procedure to obtain the equations for a spatially-continuous area-averaged mean wind and turbulence field. In this seminal work, Wilson and Shaw (1977) demonstrated how additional source and sink terms representing the flow interaction with the canopy elements emerge naturally by application of a particular spatial averaging procedure to the Reynolds-averaged Navier-Stokes equations that obtain at every point in the canopy airspace. This procedure was further developed by Raupach and Shaw (1982) for the case of a horizontal plane averaging operation. In particular, Raupach and Shaw (1982) discuss two different options for averaging over a horizontal plane; namely, horizontally averaging the equations of motion at a single time instant over a plane extensive enough "to eliminate variations due to canopy structure and the largest length scales of the turbulent flow" (scheme I) and conventional time averaging of the equations of motion followed by horizontal averaging over a plane large enough "to eliminate variations in the canopy structure" (scheme II). Scheme I has rather limited applicability since it cannot be applied to horizontally inhomogeneous canopies.

Finnigan (1985) and Raupach et al. (1986) investigated the volume-averaging method. Finnigan (1985) considered details such as plant motion (e.g., coherently waving plant canopies) which gives rise to a 'waving production' term in the transport equations for turbulence quantities. We note that plant motion is not a factor directly pertinent to the present work which focusses on urban canopy flows, but these concepts may have a bearing on the case of moving obstacles (e.g., vehicles) within the urban canopy. Following ideas of Hanjalic et al. (1980) and parallelling Shaw and Seginer (1985), Wilson (1988) developed an empirical two-band model for the turbulence kinetic energy (TKE) which represented the large- and fine-scale components of the turbulence and their dynamics [the multiple time-scale approach has seen much subsequent use (e.g., Schiestel, 1987), but parameterizing the exchange of kinetic energy between the spectral bands is a preeminent difficulty of the approach]. Here, the turbulence kinetic energy was separated into two wave-bands, corresponding to shear kinetic energy (SKE, low-frequency band) and wake kinetic energy (WKE, high-frequency band), with separate equations developed to represent their dynamics. Wilson (1985), Green (1992), Wang and Takle (1995a), Wang and Takle (1995b), Liu et al. (1996), Ayotte et al. (1999), Sanz (2003), and Wilson and Yee (2003) investigated various modifications of the k-ϵ model or the Reynolds stress transport model to account for interaction of the air with canopy elements.

Here, we present details of the mathematical framework required to derive the transport equation for the time average of the locally-spatially-averaged velocity through a building array (which is treated here as a porous medium), and the two additional prognostic equations required to close this equation set. These additional equations predict the time-averaged resolved-scale kinetic energy of turbulence, $\overline{\kappa}$, and its dissipation rate, ϵ. The closure problem relating to the 'correct' representation of the additional source/sink terms in the transport equations for mean momentum, turbulence energy, and dissipation rate is investigated in detail. Most of the work reported is motivated by conceptual and logical difficulties in the self-consistent treatment of source and sink terms in the transport equations for turbulence kinetic energy and its dissipation rate. To this end, we attempt to lay the foundations for a systematic mathematical formulation that could be used to construct the additional source/sink terms in the transport equations for $\overline{\kappa}$ and ϵ, in response (and to some extent, contradiction) to the assertions made by Wilson and Mooney (1997) that it is "impossible to know the 'correct' influence of the unresolved processes at the fence on TKE and its dissipation rate" and by Wilson et al. (1998) that "k-ϵ closures give predictions that are sensitive to details of ambiguous choices".

2. Spatial and Time Averaging Operations

Before we begin, we present a short note on the notation that will be used. The following derivations will invariably use the flexibility of the Cartesian tensor notation, with Roman indices such as i, j, or k taking values of 1, 2, or 3. We shall also employ the Einstein summation convention in which repeated indices are summed. For any flow variable ϕ, $\langle\phi\rangle$ will denote the spatial (volume) average, $\overline{\phi}$ the time average, ϕ' the departure of ϕ from its time-averaged value, and ϕ'' the departure of ϕ from its spatially-averaged value. In addition, u_i is the total velocity in the x_i-direction, with $i = 1$, 2, or 3 representing the streamwise x, spanwise y, or vertical z directions. Finally, $\mathbf{x} \equiv (x, y, z)$, $(u_1, u_2, u_3) \equiv (u, v, w)$, and t denotes time.

The derivation of a model for the spatially-averaged time-mean flow can start either from applying the spatial averaging operation to the time-averaged Navier-Stokes (NS) equation ($\langle \overline{\text{NS}} \rangle$), or the time averaging operation to the spatially-averaged Navier-Stokes equation ($\overline{\langle \text{NS} \rangle}$). Also, the concept of spatial filtering is fundamental to large-eddy simulation. The reader is referred to Ghosal and Moin (1995) and Vasilyev et al. (1998) for the application of spatial filtering in the context of large-eddy simulation. In all these applications, to formulate the 'simplified' equations of motion, we must choose a suitable decomposition of a flow property into its rapidly and slowly varying components and determine a strategy for applying the corresponding averaging operation. First, consider spatial averaging of the flow in some multiply connected space (viz., space in which not every closed path or contour within the space [set] is contractable to a point). In a "slow + fast" decomposition[2] of a flow property ϕ based on spatial filtering, scales are separated by applying a low-pass scale filter to give a filtered quantity $\langle \phi \rangle$ defined by

$$\langle \phi \rangle(\mathbf{x}) = \int_{\text{a.s.}} G(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y}) \, d\mathbf{y} \equiv G \star \phi. \qquad (1)$$

The integral in Equation (1) is assumed to be over all space (a.s.). Here, $G(\mathbf{x} - \mathbf{y})$, the convolution filter kernel, is a localized function (i.e., $G \to 0$ as $\|\mathbf{x} - \mathbf{y}\| \to \infty$, where $\| \cdot \|$ denotes the Euclidean norm) with filter width Δ which is related to some cutoff scale in space, and $\star$ is used to denote the convolution operation. In general, the filter width can depend on $\mathbf{x}$, which we will explicitly indicate using the notation $G(\mathbf{x} - \mathbf{y} | \Delta(\mathbf{x}))$.

If we assume that G is a symmetric function of $\mathbf{x} - \mathbf{y}$, and differentiate Equation (1) with respect to x_i, we get the following relationship between the spatial average of the spatial derivative and the spatial derivative of the spatial average of a quantity ϕ which could be a scalar or component of a vector (on application of the Gauss divergence

[2] The difficulty of this approach is that for fluid turbulence in plant or urban canopies, there is no real (unambiguous) separation between the large (slow) and small (fast) scales of motion, so the "slow + fast" decomposition referred to here should perhaps be interpreted more correctly as merely a convention.

theorem):

$$\left\langle \frac{\partial \phi}{\partial x_i} \right\rangle - \frac{\partial \langle \phi \rangle}{\partial x_i} \equiv \left[G\star, \frac{\partial}{\partial x_i} \right] \phi = -\left(\frac{\partial G}{\partial \Delta} \star \phi \right) \frac{\partial \Delta}{\partial x_i} + \int_S G(\mathbf{x} - \mathbf{y} | \Delta(\mathbf{x})) \phi(\mathbf{y}) n_i \, dS, \quad (2)$$

where S denotes the sum of all obstacle surfaces contained in the multiply connected region (extending over all space), n_i is the unit outward normal in the i-th direction on the surface S (positive when directed into the obstacle surface), and $[f, g] \equiv fg - gf$ denotes the commutator bracket of two operators f and g. Equation (2) will be referred to as the generalized spatial averaging theorem. A special case of this theorem (known as the spatial averaging theorem) has been derived by Raupach and Shaw (1982) and Howes and Whitaker (1985).

The spatial filtering operation does not commute with spatial differentiation. The non-commutation of these two operations results from two contributions. The first contribution is encapsulated in the first term on the right-hand-side of Equation (2) which arises from the spatial variation in filter cutoff length $\Delta(\mathbf{x})$. The second contribution, summarized in the second term on the right-hand-side of Equation (2), is due to the presence of obstacle surfaces in the multiply connected flow domain. Interestingly, if we apply the spatial-averaging operator to the continuity equation, the spatial variation in the filter width implies that $\langle u_i \rangle$ is no longer solenoidal. More specifically, although the velocity across the air/solid boundaries vanishes owing to the no-slip and impermeability boundary conditions here, the spatial variation of the filter width implies an extra source/sink term in the filtered continuity equation [which is a direct consequence of Equation (2)]:

$$\frac{\partial \langle u_i \rangle}{\partial x_i} = -\left[G\star, \frac{\partial}{\partial x_i} \right] u_i = \left(\frac{\partial G}{\partial \Delta} \star u_i \right) \frac{\partial \Delta}{\partial x_i} \neq 0, \quad (3)$$

since $\Delta = \Delta(\mathbf{x})$.

To ensure that the spatially-averaged velocity field is solenoidal, we consider a special convolution kernel whose filter cutoff length does not depend on $\mathbf{x}$. To this purpose, consider the box or top-hat filter defined as (although any other filter function with a *fixed* filter width would have been suitable also)

$$G(\mathbf{x} - \mathbf{y}) = \begin{cases} 1/\mathrm{V}, & \text{if } |x_i - y_i| < \Delta_i/2; \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Here, $V = \Delta_1\Delta_2\Delta_3 \equiv \Delta_x\Delta_y\Delta_z$ is the *constant* volume over which we average to obtain continuous variables. With the constant width filter kernel of Equation (4), the spatial average of a flow property ϕ of Equation (1) becomes simply

$$\langle\phi\rangle(\mathbf{x},t) = \frac{1}{V}\int_V \phi\, dV \equiv \frac{1}{V}\int_V \phi(\mathbf{x}+\mathbf{r},t)\, d\mathbf{r}. \qquad (5)$$

Note that in Equation (5), the averaging volume includes *both* fluid and solid parts (obstacles) [viz., the integral of ϕ is extended over the entire averaging volume and divided by its measure]. Hence, in the spatial average of Equation (5), the value of ϕ is averaged over both the fluid and solid parts with the implicit assumption that ϕ vanishes identically inside the solid parts.[3] Applying the volume-averaging operator of Equation (5) to the continuity equation results in a spatially-averaged velocity field $\langle u_i\rangle$ that is solenoidal.

We will use the spatial-averaging operation displayed in Equation (5), where the average is taken over both the fluid and solid phases in V (averaging volume), and the normalizing factor is the total volume V. For two-phase systems, two other definitions for averaging have been proposed (e.g., Miguel et al., 2001). In a two-phase system, the total averaging volume V is made up of the volume of the fluid phase V_f and the solid phase V_s, so $V = V_f + V_s$. The superficial (external) phase average of ϕ is defined as

$$\langle\phi\rangle^e = \frac{1}{V}\int_{V_f} \phi\, dV \qquad (6)$$

and the intrinsic (internal) phase average of ϕ is defined as

$$\langle\phi\rangle^i = \frac{1}{V_f}\int_{V_f} \phi\, dV. \qquad (7)$$

The intrinsic phase average is an average of a flow property over the fluid phase (i.e., the averaging volume V_f excludes the solid phase, with the normalizing factor being V_f). On the other hand, the external phase

[3] It could be argued that ϕ is strictly undefined within the solid parts (phase) of the averaging volume V and, hence, the averaging volume V should strictly extend over the fluid part (airspace) only, with the solid parts being excluded. This is, in effect, the external or internal phase average of ϕ defined later in Equations (6) and (7), respectively. Adopting the convention that ϕ vanishes identically within the solid parts of the averaging volume, it can be seen that $\langle\phi\rangle = \langle\phi\rangle^e$ (viz., the spatial average of ϕ coincides exactly with the external phase average of ϕ). However, it is important to note that $\langle\langle\phi\rangle\rangle \neq \langle\langle\phi\rangle^e\rangle^e$ owing to the fact that the external phase average does not follow the usual rules for Reynolds averaging, whereas the simpler spatial average defined by Equation (5) does.

average is a weighted average of the fluid property over the fluid phase (i.e., the total volume V is used as the normalizing factor, but the averaging excludes the solid phase). If we define the porosity (volume fraction of the fluid phase) as $\xi \equiv V_f/V$, then it can be seen the intrinsic and external phase averages of ϕ are related as $\langle\phi\rangle^e = \xi\langle\phi\rangle^i$.

The spatial (or, volume) average defined in Equation (5) seems natural in the present context, and leads to the *simplest* forms for the volume-averaged transport equations on application of the volume-averaging operator to the continuity and the Reynolds equation for mean momentum at a single point. Although V is a constant that is independent of the spatial coordinates, V_f which represents the volume of the fluid phase contained within V need not be (e.g., for an inhomogeneous canopy, V_f and V_s will be a function of the spatial coordinates). Because V_f depends on $\mathbf{x}$ in Equation (7) (i.e., the filter width depends on $\mathbf{x}$), the use of the intrinsic phase average will result in a filtered velocity that is not solenoidal [cf. Equation (3)]. Even so, transport equations for the intrinsic phase-averaged velocity $\langle\bar{u}_j\rangle^i$ can be derived, provided that the fluid-phase volume $V_f(\mathbf{x})$ is differentiable (or, equivalently, that the porosity $\xi \equiv V_f/V$ is a differentiable function of $\mathbf{x}$) although this will result in a number of 'extra' source/sink terms in these equations arising solely from the dependence of V_f on the spatial coordinates.[4] These 'extra' source/sink terms, arising from the spatial variation of the volume fraction (porosity), will necessarily complicate the equations of motions based on the intrinsic phase average in comparison with those based on the simple spatial (volume) average of Equation (5). In particular, setting $\phi \equiv 1$ in Equation (2) and assuming that G is the top-hat filter exhibited in Equation (4), we obtain the result that

$$\frac{\partial \xi}{\partial x_i} = -\frac{1}{V}\int_S n_i \, dS. \tag{8}$$

In deriving Equation (8), we used the fact that $\langle 1\rangle = \langle 1\rangle^e = \xi\langle 1\rangle^i = \xi$. Hence, the volume-averaged transport equations for $\langle\phi\rangle^i$ ($\phi \equiv \bar{u}_j$, $j =$

[4] Spatial variation in the filter width (e.g., $\Delta(\mathbf{x}) = V_f^{1/3}(\mathbf{x})$ as used in the intrinsic phase averaging operation) implies additional energy transfer mechanisms (both local energy drain and backscatter) between the resolved and sub-filter scales of the flow, resulting in the 'extra' source/sink terms alluded to here. These additional sources of local energy drain and backscatter will depend on the specific local filter-width variations (or, equivalently, on the porosity in our case). Indeed, these local filter width variations could lead to the apparent destruction of a local resolved flow structure as it is advected by the flow from a region of small filter width to one with large filter width (previously resolved-scale flow feature now becomes a sub-filter scale flow feature); or, alternatively, to the apparent 'spontaneous' emergence of a resolved-scale flow pattern from a collection of sub-filter scale patterns as these patterns are advected from a region of large filter width to one of small filter width.

$1, 2, 3$) will necessarily include additional source/sink terms involving surface integrals of the form of Equation (8) devolving solely from the spatial variation in the porosity ξ.

The external phase average does not seem natural in the present context because $\langle\langle\phi\rangle^e\rangle^e \neq \langle\phi\rangle^e$ (or, equivalently, $\langle\phi_e''\rangle^e \neq 0$ where $\phi_e'' \equiv \phi - \langle\phi\rangle^e$ [viz., deviation of ϕ from its external phase average value]).[5] In other words, $\langle\cdot\rangle^e$ is difficult at worst (inconvenient at best) to work with, since this averaging operator is not a Reynolds operator satisfying the usual rules for Reynolds averaging.

In analogy with the spatial (or, volume) average defined in Equation (5), the time average[6] of ϕ, which we denote using an overbar, will be defined as

$$\bar{\phi}(\mathbf{x}) = \frac{1}{T}\int_{t_0}^{t_0+T} \phi(\mathbf{x}, t)\, dt, \qquad T_1 \ll T \ll T_2. \tag{9}$$

In view of Equations (5) and (9), time and spatial averaging commute so $\overline{\langle\phi\rangle} = \langle\overline{\phi}\rangle$. In Equation (5), the horizontal averaging scales Δ_x, Δ_y need to be large compared to the separation between individual roughness elements, but much less than the characteristic length scales over which the density of the roughness elements changes; but to ensure a sufficient vertical resolution of the flow property gradients, $\Delta_z \ll \Delta_x, \Delta_y$ making V a thin, horizontal slab. In Equation (9), the averaging time T is implicitly assumed to be sufficiently long to ensure that many cycles of the rapid turbulent fluctuations in a flow property are captured, but sufficiently short so that the external large-scale variations in the flow property are approximately constant. Hence, in Equation (9), T_1 and T_2 denote the time scales characteristic of the rapid and slow variations in the flow property ϕ, with the implicit assumption that T_1 and T_2 differ by several orders of magnitude.

[5] Note that after spatial averaging, a flow property ϕ is a continuous function of the coordinates in a multiply connected space. Hence, even though ϕ vanishes identically (or, alternatively, is undefined) in the solid phase within V, its spatially-averaged value $\langle\phi\rangle^e$ is continuous and nonzero (and, consequently, well-defined) in the entire averaging volume V, so $\langle\langle\phi\rangle^e\rangle^e \neq \langle\phi\rangle^e$.

[6] In this paper, we assume implicitly that the meteorological variables are described (approximately or better) by a stationary random process, for which the time average can be meaningfully defined. For a non-stationary random process, it is necessary to replace the time average used here with the ensemble average (viz., the average of a quantity ϕ, as a function of space $\mathbf{x}$ and time t, over an "ensemble" of realizations of ϕ measured under a particular set of mean weather conditions). In view of this, the time-averaging operation used in this paper can be replaced by the ensemble-averaging operation, so that it is correct henceforth to interpret $\overline{\phi}$ as either the time- or ensemble-average of ϕ.

In general, ϕ can be decomposed in the following two ways:

$$\phi = \bar{\phi} + \phi', \qquad \overline{\phi'} = 0, \tag{10}$$

or

$$\phi = \langle \phi \rangle + \phi'', \qquad \langle \phi'' \rangle = 0. \tag{11}$$

Although $\langle \overline{\mathrm{NS}} \rangle = \overline{\langle \mathrm{NS} \rangle}$ owing to the commutation of time and spatial averaging operations, space-time filtering and time-space filtering of the Navier-Stokes equation lead to two different decompositions for the turbulent stress tensor, a quantity that needs to be modelled (viz., the turbulence closure problem). The subtle (but important) differences in these two decompositions for the turbulent stress tensor (arising from either a space-time or time-space filtering of the nonlinear convective term in the Navier-Stokes equation) will be elucidated in the following sections.

3. Spatial Average of the Time-Averaged NS Equation

The spatial average of the time-averaged NS equation (or spatial average of the RANS equation) has been described in detail by Raupach and Shaw (1982) as their scheme II, Raupach et al. (1986), Ayotte et al. (1999), and others. Consequently, only some final results are summarized here for later reference. The spatially-averaged RANS equation for the prediction of the spatially-averaged time-mean velocity $\langle \overline{u_i} \rangle$ is

$$\frac{\partial \langle \bar{u}_i \rangle}{\partial t} + \frac{\partial \langle \bar{u}_j \rangle \langle \bar{u}_i \rangle}{\partial x_j} = -\frac{\partial \langle \bar{p} \rangle}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \bar{f}_i, \tag{12}$$

with

$$\tau_{ij} \equiv -\langle \overline{u_i' u_j'} \rangle - \langle \bar{u}_i'' \bar{u}_j'' \rangle + \nu \frac{\partial \langle \bar{u}_i \rangle}{\partial x_j}, \tag{13}$$

and

$$\bar{f}_i = \underbrace{\frac{\nu}{V} \int_S \frac{\partial \bar{u}_i}{\partial n} dS}_{\text{viscous drag}} - \underbrace{\frac{1}{V} \int_S \bar{p} n_i dS}_{\text{form drag}}. \tag{14}$$

Here, $\bar{p}$ is the kinematic mean pressure, $\bar{f}_i$ is the total mean drag force per unit mass of air in the averaging volume composed of the sum of a form (pressure) drag and a viscous drag, and τ_{ij} is the spatially-averaged kinematic total stress tensor. In Equation (14), ν is the kinematic viscosity, S is the part of the bounding surface *inside* the averaging volume V that coincides with the obstacle surfaces, n_i is a unit normal vector in the i-th direction pointing from

V into S (viz., directed from the fluid into the solid surface), and $\partial/\partial n$ denotes differentiation along the direction normal to the surface S. The lack of commutativity between filtering and spatial differentiation [cf. Equation (2)] causes every spatial derivative operator in the Reynolds-averaged Navier-Stokes equation to generate terms that cannot be expressed solely in terms of the time-mean, spatially-averaged velocity fields. In consequence, a "closure problem" is introduced not only for the nonlinear convective term, but also for some of the linear terms as well (e.g., the mean pressure gradient and viscous diffusion terms give rise to the additional form and viscous drag terms on spatial averaging). Finally, we note that the spatially-averaged kinematic Reynolds stresses $\langle\overline{u_i'u_j'}\rangle$ and kinematic dispersive stresses $\langle\bar{u}_i''\bar{u}_j''\rangle$ are a direct consequence of the spatial averaging of the time-averaged nonlinear convective term u_iu_j; viz.,

$$\langle\overline{u_iu_j}\rangle = \langle\bar{u}_i\rangle\langle\bar{u}_j\rangle + \langle\overline{u_i'u_j'}\rangle + \langle\bar{u}_i''\bar{u}_j''\rangle. \quad (15)$$

The last term on the right-hand-side of Equation (15) is the dispersive stress which arises from the spatial correlation in the time-mean velocity field varying with position in the averaging volume V. Although Ayotte et al. (1999) rigorously derive the transport equation for $\langle\overline{u_i'u_j'}\rangle$, the extra source/sink terms in their proposed model for the spatially-averaged second central velocity moments $\overline{u_i'u_j'}$, which they denote as d_{ij} (contribution to the total dissipation arising from the canopy interaction processes), were obtained from an approximate expression for the work done by the fluctuating turbulence against the fluctuating drag force. The latter was derived in the context of the time average of the spatially-averaged NS equation. Mixing the spatially-averaged RANS formulation with the time-averaged, spatially-averaged NS formulation results in a mathematical inconsistency in the approach described by Ayotte et al. (1999).

In the next section, we formulate the equation set for the time-averaged, spatially-averaged NS approach. The approach taken here is similar to that proposed by Wang and Takle (1995a) and Getachew et al. (2000).

4. Time Average of Spatially-Averaged NS Equation

The spatial average of the nonlinear convective term u_iu_j in the Navier-Stokes equation can be expanded as follows:

$$\langle u_iu_j\rangle = \Big\langle\, (\langle u_i\rangle + u_i'')(\langle u_j\rangle + u_j'')\,\Big\rangle = \langle u_i\rangle\langle u_j\rangle + \langle u_i''u_j''\rangle. \quad (16)$$

Using the decomposition for the spatially-averaged nonlinear convective term of Equation (16), and applying the spatial averaging theorem in Equation (2) with the filter kernel defined in Equation (4), the spatially-averaged NS equation assumes the form

$$\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial \langle u_j \rangle \langle u_i \rangle}{\partial x_j} = -\frac{\partial \langle p \rangle}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + f_i, \tag{17}$$

where

$$T_{ij} = -\langle u_i'' u_j'' \rangle + \nu \frac{\partial \langle u_i \rangle}{\partial x_j}, \tag{18}$$

and

$$f_i = \frac{\nu}{V} \int_S \frac{\partial u_i}{\partial n} dS - \frac{1}{V} \int_S p n_i dS. \tag{19}$$

We remind the reader that the application of local spatial-averaging to u_i to give $\langle u_i \rangle$ smooths (attenuates) the fluctuations at scales below the filter width Δ, but that the spatially-averaged velocity $\langle u_i \rangle$ still exhibits turbulent fluctuations at scales larger than Δ. However, $\langle u_i \rangle$ is now a turbulent quantity that is continuous in space.

Time-averaging Equation (17) gives the time-averaged, spatially-averaged Navier-Stokes equation $\overline{\langle \mathrm{NS} \rangle}$ as

$$\frac{\partial \langle \bar{u}_i \rangle}{\partial t} + \frac{\partial \langle \bar{u}_j \rangle \langle \bar{u}_i \rangle}{\partial x_j} = -\frac{\partial \langle \bar{p} \rangle}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \bar{f}_i, \tag{20}$$

where

$$\tau_{ij} \equiv -\overline{\langle u_i' \rangle \langle u_j' \rangle} + \overline{T_{ij}} = -\overline{\langle u_i' \rangle \langle u_j' \rangle} - \langle \overline{u_i'' u_j''} \rangle + \nu \frac{\partial \langle \bar{u}_i \rangle}{\partial x_j}, \tag{21}$$

and $\bar{f}_i$ here is the same as $\bar{f}_i$ defined in Equation (14).

From Equations (12), (13), (20) and (21), the following relationship holds:

$$\langle \overline{u_i' u_j'} \rangle + \langle \bar{u}_i'' \bar{u}_j'' \rangle = \overline{\langle u_i' \rangle \langle u_j' \rangle} + \langle \overline{u_i'' u_j''} \rangle. \tag{22}$$

Equation (22) is the necessary and sufficient condition for $\langle \overline{\mathrm{NS}} \rangle = \overline{\langle \mathrm{NS} \rangle}$. The total stress tensors τ_{ij}, defined in either Equation (13) or Equation (21) are identical (hence, the same notation is used here for these two quantities), although the individual terms in their sums are different. We note that the physical character of the term $\langle \overline{u_i'' u_j''} \rangle$ is different from the conventional dispersive term $\langle \bar{u}_i'' \bar{u}_j'' \rangle$. The conventional dispersive stresses $\langle \bar{u}_i'' \bar{u}_j'' \rangle$ in the spatially-averaged RANS equation correspond to stresses arising from correlations in the point-to-point variations in the time-averaged (mean) velocity field. However, the 'dispersive' flux term $\langle \overline{u_i'' u_j''} \rangle$ in the time-average of the spatially-averaged

NS equation corresponds to stresses arising from correlations (obtained using space-time averaging) of large wavenumber (or frequency) velocity fluctuations $u_i'' \equiv (1 - G) \star u_i$ where 1 is the identity operator with respect to the convolution operation (which, in this case should be interpreted as the Dirac delta function) and G is the top-hat filter defined in Equation (4) (viz., correlation in the sub-filter motions with characteristic wavenumber greater than $\approx \pi/\Delta$, where $\Delta \equiv V^{1/3}$ is the width of the spatial filter).[7] Alternatively, this dispersive stress term can be described simply as the time average of the local-scale spatial covariance $\langle u_i'' u_j'' \rangle$ of two components of the local spatial anomaly in velocity. It is not clear *a priori* how $\langle \overline{u_i'' u_j''} \rangle$ is related to $\langle \bar{u}_i'' \bar{u}_j'' \rangle$.

From the turbulence modelling point of view, in this paper and III we will model $\overline{\langle u_i' \rangle \langle u_j' \rangle}$ in Equation (21) using the Boussinesq eddy viscosity (ν_t) closure as follows:

$$\overline{\langle u_i' \rangle \langle u_j' \rangle} = \frac{2}{3}\delta_{ij}\overline{\kappa} - \nu_t \left(\frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} + \frac{\partial \langle \bar{u}_j \rangle}{\partial x_i} \right). \tag{23}$$

Here $\overline{\kappa}$ and ν_t are defined as

$$\overline{\kappa} \equiv \frac{1}{2}\overline{\langle u_i' \rangle \langle u_i' \rangle}, \qquad \nu_t = C_\mu \frac{\overline{\kappa}^2}{\epsilon}, \tag{24}$$

and ϵ is defined as

$$\epsilon \equiv \nu \overline{\frac{\partial \langle u_i' \rangle}{\partial x_k} \frac{\partial \langle u_i' \rangle}{\partial x_k}}. \tag{25}$$

In Equation (24), C_μ is a closure (empirical) constant taken to be 0.09 as in the standard k-ϵ model for turbulence closure (Launder and Spalding, 1974). We note that κ is the resolved-scale kinetic energy of turbulence (viz., κ embodies the energetics of turbulent flow in the resolved scales of motion with wavenumber less than $\approx \pi/\Delta$), so $\overline{\kappa}$ is the time-averaged resolved-scale turbulence kinetic energy. In addition, since the filter in Equation (5) is positive (volume averaging), κ is necessarily a non-negative definite quantity.

To make further progress, we assume

$$\langle \overline{u_i'' u_j''} \rangle \ll \overline{\langle u_i' \rangle \langle u_j' \rangle} \tag{26}$$

in the present study (viz., we will simply neglect the term $\langle \overline{u_i'' u_j''} \rangle$ for expediency since no reference data exists at this time to guide its modelling). Even so, it is possible to construct a structural model for the

[7] Recall that spatial filtering of the instantanenous velocity field using Equation (2) introduces a length scale into the description of the fluid dynamics, namely the width $\Delta \equiv V^{1/3}$ of the filter used.

'conventional' dispersive stress tensor (see Appendix A) by applying a regularization to the nonlinear convective term in the RANS equation. Before we derive the model equations for $\overline{\kappa}$ and ϵ, let us examine the momentum sink f_i (arising from the pressure and viscous forces created by the obstacle elements in V) in the spatially-averaged NS equation [cf. Equations (17) and (19)]. We require a parameterization for f_i, which is the drag force exerted by the obstacle elements on a unit mass of air in V.

To proceed, we note that heuristic correlations of experimental data for flow through a porous medium (Scheidegger, 1974) show that at low speeds the pressure drop (drag force) caused by viscous drag is directly proportional to the (volume-averaged) velocity (Darcy's law). However, this relationship needs to be modified at higher velocities to account for an increase in the form drag (inertial effects). Experimental observations indicate that the pressure drop (drag force) for a general flow through a porous medium is proportional to a linear combination of flow velocity (arising from viscous resistance due to the obstacle boundaries) and the square of flow velocity (arising from resistance due to the inertial forces).[8] For the current application, where the obstacle elements are unresolved and the aggregation of obstacle elements in the urban canopy is treated simply as a porous medium, this phenomenologically-based law can be formulated quantitatively in terms of the following

[8] Experimental observations indicate that the pressure drop (or drag force) for uni-directional flow in the x-direction in the bulk of a porous medium is described quantitatively as

$$f_x \equiv \frac{\delta P}{\delta x} = -\frac{\nu}{K_p}\langle u\rangle - C_D\hat{A}\langle u\rangle^2,$$

where P is the kinematic pressure (pressure normalized by fluid density), ν is the kinematic fluid viscosity, K_p is the permeability of the porous medium, C_D is the drag coefficient, $\hat{A}$ is the frontal area density, and $\langle u\rangle$ is the spatially-averaged (volume-averaged, or Darcian) velocity in the x-direction. In accordance with meteorological convention, we have used the drag parameter $C_D\hat{A}$ to parameterize the inertial effects, rather than following the porous media convention (Scheidegger, 1974) where the drag parameter is replaced by $F_r/K_p^{1/2}$ where F_r is the Forchheimer constant (so, the identification $C_D\hat{A} \leftrightarrow F_r/K_p^{1/2}$ can be used to convert between these two conventions). Finally, this phenomenological relationship for the drag force exerted by the porous medium on the flow, which is specific for flow in the x-direction, can be generalized to the tensorial representation of Equation (27) by re-casting it in terms of the spatially-averaged velocity vector $\langle u_i\rangle$ and the scalar (invariant) $\langle u_i\rangle\langle u_i\rangle$ to give a form for the relationship that is valid in an arbitrary inertial frame of reference.

covariant tensorial representation (for non-Darcian flow):

$$f_i = \underbrace{-\frac{\nu}{K_p}\langle u_i \rangle}_{f_{V\,i}} - \underbrace{C_D \hat{A}\big(\langle u_j \rangle \langle u_j \rangle\big)^{1/2} \langle u_i \rangle}_{f_{F\,i}}, \tag{27}$$

where ν is the kinematic fluid viscosity, K_p the permeability, C_D the element drag coefficient, and $\hat{A}$ the frontal area density (frontal area of obstacles exposed to the wind per unit volume). In Equation (27), $f_{V\,i}$ and $f_{F\,i}$ refer to the viscous and form drag contributions, respectively, to the total drag term.

The drag coefficient C_D depends on the permeability K_p, the Reynolds number $Re_{K_p} = \mathcal{U}\sqrt{K_p}/\nu$ ($\mathcal{U} \equiv (\langle u_i \rangle \langle u_i \rangle)^{1/2}$ is a characteristic local velocity) and the microstructure of the porous medium (or, equivalently, the geometrical and topological structure of the plant or urban canopy). Indeed, C_D through its functional dependence on K_p must depend on the porosity and on the overall morphology of the pore space (air space) between the canopy elements and, as such, is a phenomenological parameter that must encapsulate, in an averaged sense, the effects of the obstacles on producing the complex point-to-point variations in fluid motions that take place in the air space between the canopy elements. For simplicity, we assume that $f_{V\,i} \ll f_{F\,i}$ in Equation (27), implying that the drag force term can be parameterized using the following common formulation

$$f_i = -C_D \hat{A}\big(\langle u_j \rangle \langle u_j \rangle\big)^{1/2} \langle u_i \rangle. \tag{28}$$

It will be shown in III that the assumption $f_{V\,i} \ll f_{F\,i}$ is valid for a coarse-scaled cuboid obstacle array.[9] The diagnosis of the drag coefficient C_D as a function of position (x, z) within the urban canopy for a spatially developing flow through the canopy will be undertaken in III.

Following from this parameterization, $\bar{f}_i$ (time-averaged momentum sink) in Equation (14) is required to be modelled (approximated) as

$$\bar{f}_i = -C_D \hat{A}\overline{\big(\langle u_j \rangle \langle u_j \rangle\big)^{1/2} \langle u_i \rangle}. \tag{29}$$

Substituting $\langle u_i \rangle = \langle \bar{u}_i \rangle + \langle u_i' \rangle$ into Equation (28) and using the binomial series to approximate the square root term up to second order in

[9] Equation (28) is the conventional parameterization for the drag force used in modelling fine-scaled plant canopy flows, but it should be emphasized that in this application the approximation that $f_{V\,i} \ll f_{F\,i}$ is probably poorer than in the case of coarse-scaled urban canopy flows (especially deep in the plant canopy where wind speeds are low). For example, Thom (1968) reported that the ratio of form to viscous drag (skin friction) was about 3 to 1 for bean leaves at typical wind incidence angles.

the velocity fluctuations $\langle u_i' \rangle$ (or, equivalently, truncation of the Taylor series expansion of the square root term at second order in the expansion (order) parameter $\delta \equiv |\langle u_i' \rangle| / \big(\langle \bar{u}_j \rangle \langle \bar{u}_j \rangle\big)^{1/2}$),[10] an approximate form for f_i that is appropriate for time averaging can be derived as

$$f_i \approx -C_D \hat{A} Q \Bigg(\langle \bar{u}_i \rangle + \frac{\langle \bar{u}_i \rangle \langle \bar{u}_j \rangle \langle u_j' \rangle}{Q^2} + \langle u_i' \rangle + \frac{\langle u_i' \rangle \langle \bar{u}_j \rangle \langle u_j' \rangle}{Q^2} + \frac{\langle \bar{u}_i \rangle \langle u_j' \rangle \langle u_j' \rangle}{2Q^2} \Bigg) + O(\delta^3), \quad (30)$$

where $Q \equiv \big(\langle \bar{u}_i \rangle \langle \bar{u}_i \rangle\big)^{1/2}$ is the magnitude of the spatially-averaged, time-mean wind speed.[11]

The approximation for f_i in Equation (30) involves an orderly expansion about a local, volume-averaged mean flow state $\langle \bar{u}_i \rangle$. The expansion about this mean flow state suggests that the ratio of the turbulence kinetic energy to the mean flow kinetic energy is small

[10] Strictly speaking, absolute convergence of the binomial series for the square root term requires that

$$\left| \frac{\langle \bar{u}_j \rangle \langle u_j' \rangle}{Q^2} + \frac{\kappa}{Q^2} \right| < \frac{1}{2},$$

where $Q \equiv \big(\langle \bar{u}_i \rangle \langle \bar{u}_i \rangle\big)^{1/2}$. However, even if this condition is not satisfied, it is nevertheless possible for the first few terms in the expansion of $\big(\langle u_j \rangle \langle u_j \rangle\big)^{1/2}$ in the putative small (order) parameter δ to be useful (viz., provide useful predictions) even though the expansion parameter is of order unity (which, strictly, will result in the violation of the condition for absolute convergence of the binomial series). There are numerous examples of this phenomenon in other fields of endeavor: in quantum chromodynamics (QCD), the coupling constant for the strong interaction (force which holds the nucleus together) is large ($\alpha_{\text{strong}} \sim 14$) and the power expansion in the order parameter α_{strong} (perturbation theory) is not expected to be reliable, yet the first few terms in perturbative calculations in QCD have been found surprisingly to give predictions that agree well with experimental results in deep electron-nucleon scattering reactions (Greiner and Schäfer, 1994); and, the expansion in small Reynolds number parameter for laminar flow around a cylinder is known to work well for Reynolds number of order 10 (Van Dyke, 1964).

[11] An alternative procedure would have been to use the full infinite series (formal) expansion for the square root term, and then to assume Gaussian turbulence in the canopy flow so that Wick's theorem (Kleinert, 1990) can be applied to the time-averaged expansion to express all the higher-order velocity moments in terms of only various products of the second-order velocity moments. This produces an exact formal (albeit necessarily complicated) result for $\bar{f}_i$ for the case of Gaussian turbulence, but is physically deficient since canopy turbulence is known to be strongly non-Gaussian (Raupach et al., 1986). Furthermore, for strong turbulence where $\delta = |\langle u_i' \rangle| / Q \sim O(1)$, the formal expansion of the square root term may not result in a convergent series. Yet, the first few terms in this expansion may still nevertheless lead to useful results (see footnote 10).

(viz., $\langle u_i' \rangle \langle u_i' \rangle / \langle \bar{u}_i \rangle \langle \bar{u}_i \rangle \ll 1$; or, equivalently, $\delta \ll 1$). However, it is known that this ratio of kinetic energies is not small in a plant or urban canopy (Raupach et al., 1986; Roth, 2000), being of order unity (strong turbulence). It is legitimate to ask why one has the right to expect the second-order expansion for f_i in Equation (30) to resemble the real drag force, when the implicit parameter of the expansion is of order unity. To this issue, we emphasize that by following a rational procedure for the physical parameterization and subsequent mathematical approximation of f_i (required for time-averaging) and for construction of the implied corresponding source/sink terms in the supporting transport equations for the turbulence quantities (see below), we have created functional forms for these terms that while not being exact, nevertheless probably mimic the real behavior (and essential physics) of these terms to a good approximation. In the end, we only claim here to have produced a self-consistent basis for the approximations used for the source/sink terms in the mean momentum and supporting turbulence transport equations. However, an alternative non-perturbative method for the evaluation of $\bar{f}_i$ that is valid for strong turbulence is summarized in Appendix B.

Time averaging of f_i in Equation (30) gives the following expression (approximation) for the time-averaged form and viscous drag force vector exerted on a unit mass of air in the averaging volume:

$$\bar{f}_i = -C_D \hat{A} \left(Q \langle \bar{u}_i \rangle + \frac{\langle \bar{u}_j \rangle}{Q} \overline{\langle u_i' \rangle \langle u_j' \rangle} + \frac{\langle \bar{u}_i \rangle \overline{\kappa}}{Q} \right), \tag{31}$$

which, in combination with Equation (23), yields

$$\bar{f}_i = -C_D \hat{A} \left[\left(Q + \frac{5}{3} \frac{\overline{\kappa}}{Q} \right) \langle \bar{u}_i \rangle - \nu_t \left(\frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} + \frac{\partial \langle \bar{u}_j \rangle}{\partial x_i} \right) \frac{\langle \bar{u}_j \rangle}{Q} \right]. \tag{32}$$

With these closure assumptions, the final form of the modelled time-averaged, spatially-averaged NS equation [obtained by substituting Equations (23), (26) and (32) into Equations (20) and (21)] becomes

$$\begin{aligned} \frac{\partial \langle \bar{u}_i \rangle}{\partial t} + \frac{\partial \langle \bar{u}_j \rangle \langle \bar{u}_i \rangle}{\partial x_j} = & -\frac{\partial \langle \bar{p} \rangle}{\partial x_i} + \frac{\partial}{\partial x_j} \left[(\nu + \nu_t) \left(\frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} + \frac{\partial \langle \bar{u}_j \rangle}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \overline{\kappa} \right] \\ & -C_D \hat{A} \left[\left(Q + \frac{5}{3} \frac{\overline{\kappa}}{Q} \right) \langle \bar{u}_i \rangle \right. \\ & \left. -\nu_t \left(\frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} + \frac{\partial \langle \bar{u}_j \rangle}{\partial x_i} \right) \frac{\langle \bar{u}_j \rangle}{Q} \right]. \end{aligned} \tag{33}$$

5. Derivation of Transport Equations for $\overline{\kappa}$ and ϵ

The budget equations for $\overline{\kappa}$ and ϵ, which have been defined explicitly in Equations (24) and (25), need to be derived. To this purpose, let us define $f_i' \equiv f_i - \bar{f}_i$ as the fluctuating drag force. From Equations (30) and (31), we can derive

$$f_i' = -C_D \hat{A} Q \left(\frac{\langle \bar{u}_i \rangle \langle \bar{u}_k \rangle \langle u_k' \rangle}{Q^2} + \langle u_i' \rangle + \frac{\langle u_i' \rangle \langle \bar{u}_k \rangle \langle u_k' \rangle}{Q^2} - \frac{\langle \bar{u}_k \rangle}{Q^2} \overline{\langle u_i' \rangle \langle u_k' \rangle} + \frac{\langle \bar{u}_i \rangle \langle u_k' \rangle \langle u_k' \rangle}{2Q^2} - \frac{\langle \bar{u}_i \rangle \overline{\kappa}}{Q^2} \right). \quad (34)$$

The transport equation for the spatially-averaged fluctuating velocity $\langle u_i' \rangle$, obtained by subtracting the evolution equation for the time-averaged spatially-averaged velocity [Equation (20)] from the spatially-averaged Navier-Stokes equation [Equation (17)], can be written in symbolic form as follows:

$$\frac{D\langle u_i' \rangle}{Dt} = \cdots + f_i', \quad (35)$$

where D/Dt is the material derivative based on the spatially-averaged velocity $\langle u_i \rangle$. The time average of the linear combination $\langle u_j' \rangle D\langle u_i' \rangle / Dt + \langle u_i' \rangle D\langle u_j' \rangle / Dt$ gives the following transport equation for $\overline{\langle u_i' \rangle \langle u_j' \rangle}$:

$$\overline{\langle u_j' \rangle \frac{D\langle u_i' \rangle}{Dt} + \langle u_i' \rangle \frac{D\langle u_j' \rangle}{Dt}} = \frac{\bar{D} \overline{\langle u_i' \rangle \langle u_j' \rangle}}{\bar{D}t} = \cdots + F_{ij}, \quad (36)$$

where $\bar{D}/\bar{D}t$ is the material derivative based on the spatially-averaged, time-mean velocity $\langle \overline{u_i} \rangle$. Furthermore, F_{ij}, representing the interaction between the fluctuating drag force and spatially-averaged velocity fluctuations, has the explicit form

$$\begin{aligned} F_{ij} &\equiv \overline{\langle u_j' \rangle f_i' + \langle u_i' \rangle f_j'} \\ &= -C_D \hat{A} \Big[2Q \overline{\langle u_i' \rangle \langle u_j' \rangle} + \frac{1}{Q} \Big(\langle \bar{u}_i \rangle \langle \bar{u}_k \rangle \overline{\langle u_j' \rangle \langle u_k' \rangle} + \langle \bar{u}_j \rangle \langle \bar{u}_k \rangle \overline{\langle u_i' \rangle \langle u_k' \rangle} \Big) \\ &\quad + \frac{2}{Q} \langle \bar{u}_k \rangle \overline{\langle u_i' \rangle \langle u_j' \rangle \langle u_k' \rangle} \\ &\quad + \frac{1}{2Q} \Big(\langle \bar{u}_i \rangle \overline{\langle u_j' \rangle \langle u_k' \rangle \langle u_k' \rangle} + \langle \bar{u}_j \rangle \overline{\langle u_i' \rangle \langle u_k' \rangle \langle u_k' \rangle} \Big) \Big]. \quad (37) \end{aligned}$$

One-half the trace of F_{ij} yields

$$\begin{aligned} F \equiv \frac{1}{2} F_{ii} &= \overline{\langle u_i' \rangle f_i'} \\ &= -C_D \hat{A} \left[2Q\overline{\kappa} + \frac{1}{Q} \Big(\langle \bar{u}_i \rangle \langle \bar{u}_k \rangle \overline{\langle u_i' \rangle \langle u_k' \rangle} \Big) \right. \\ &\qquad \left. + \frac{3}{2Q} \Big(\langle \bar{u}_k \rangle \overline{\langle u_i' \rangle \langle u_i' \rangle \langle u_k' \rangle} \Big) \right]. \end{aligned} \tag{38}$$

The triple correlation term $\overline{\langle u_i' \rangle \langle u_i' \rangle \langle u_k' \rangle}$ in Equation (38) can be modelled, following Daly and Harlow (1970), as

$$\overline{\langle u_i' \rangle \langle u_i' \rangle \langle u_k' \rangle} = 2C_s \frac{\overline{\kappa}}{\epsilon} \left[\overline{\langle u_k' \rangle \langle u_l' \rangle} \frac{\partial \overline{\kappa}}{\partial x_l} + \overline{\langle u_i' \rangle \langle u_l' \rangle} \frac{\partial \overline{\langle u_i' \rangle \langle u_k' \rangle}}{\partial x_l} \right], \tag{39}$$

where the closure constant $C_s \approx 0.3$ is used in the present study.[12] This is a gradient transport model for the third moments of the spatially-averaged fluctuating velocity and involves a tensor eddy viscosity. In Equations (38) and (39), the double correlation $\overline{\langle u_i' \rangle \langle u_j' \rangle}$ was modelled previously using the constitutive relationship in Equation (23).

The transport equation for the time-averaged, resolved-scale kinetic energy of turbulence $\overline{\kappa}$ is obtained by multiplying Equation (35) by $\langle u_i' \rangle$ and time averaging the result. This procedure will give rise to the F term exhibited in Equation (38). This term represents the interaction of the flow with the obstacle elements and corresponds explicitly to the work done by the turbulence against the fluctuating drag force. The term F can be interpreted as an additional physical mechanism for the production/dissipation of $\overline{\kappa}$ associated with work against form and viscous drag on the obstacle elements. From this perspective, the exact transport equation for $\overline{\kappa}$ is

$$\frac{\partial \overline{\kappa}}{\partial t} + \langle \bar{u}_j \rangle \frac{\partial \overline{\kappa}}{\partial x_j} = -\frac{\partial T_j}{\partial x_j} - \overline{\langle u_i' \rangle \frac{\partial}{\partial x_j} \langle u_i'' u_j'' \rangle} + (P + F) - \epsilon, \tag{40}$$

where $F \equiv \overline{\langle u_i' \rangle f_i'}$; the flux T_j is

$$T_j \equiv \frac{1}{2} \overline{\langle u_j' \rangle \langle u_i' \rangle \langle u_i' \rangle} + \overline{\langle u_j' \rangle \langle p' \rangle} - \nu \frac{\partial \overline{\kappa}}{\partial x_j}; \tag{41}$$

and,

$$P \equiv -\overline{\langle u_i' \rangle \langle u_j' \rangle} \frac{\partial \langle \bar{u}_i \rangle}{\partial x_j} \tag{42}$$

[12] Alternatively, one can assume Gaussian turbulence in the canopy flow and simply set this triple correlation (odd moment) to zero.

is the production term (which is generally positive, and hence a 'source' in the $\overline{\kappa}$ equation). In addition to F, the exact transport equation for $\overline{\kappa}$ embodies an extra term represented by the second term on the right-hand-side of Equation (40). This term is the energy redistribution due to the interaction of the spatially-averaged velocity fluctuations with the gradient of the instantaneous dispersive stresses $\langle u_i'' u_j'' \rangle$.

The modelled transport equation for $\overline{\kappa}$ is then obtained as follows. Firstly, the additional energy redistribution term identified above is assumed to be negligible, and will be ignored henceforth. Secondly, the energy flux T_j is modelled with a gradient diffusion hypothesis[13]

$$T_j = -\frac{\nu_t}{\sigma_k} \frac{\partial \overline{\kappa}}{\partial x_j}, \tag{43}$$

where the 'turbulent Prandtl number' for $\overline{\kappa}$ is assumed to be $\sigma_k = 1$. Thirdly, the additional physical effect on $\overline{\kappa}$ due to viscous and form drag on the obstacle elements embodied in $F \equiv \overline{\langle u_i' \rangle f_i'}$ will be modelled using Equations (38) and (39). With these closure approximations, the model transport equation for $\overline{\kappa}$ assumes the form

$$\frac{\partial \overline{\kappa}}{\partial t} + \frac{\partial \langle \bar{u}_j \rangle \overline{\kappa}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_k} \frac{\partial \overline{\kappa}}{\partial x_j} \right) + (P + F) - \epsilon, \tag{44}$$

where the explicit form for F is exhibited in Equations (38) and (39).

The exact transport equation for ϵ can be derived rigorously, but it is not a useful starting point for a model equation. Consequently, rather than being based on the exact equation, the model equation for ϵ here is essentially a dimensionally consistent analog to the $\overline{\kappa}$-equation. In this sense, the model equation for ϵ is best viewed as being entirely empirical. To this purpose, we note that the time scale $\tau \equiv \overline{\kappa}/\epsilon$ will make the production and dissipation terms in the $\overline{\kappa}$-equation dimensionally consistent. Indeed, $\tau = \overline{\kappa}/\epsilon$ is the only turbulence time scale that one can construct from the parameters of the problem. Hence, the dimensionally consistent and coordinate invariant analog to Equation (44) becomes

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial \langle \bar{u}_j \rangle \epsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right) + \frac{\epsilon}{\overline{\kappa}} (C_{\epsilon 1}(P + F) - C_{\epsilon 2} \epsilon), \tag{45}$$

[13] This hypothesis is consistent with the overriding (implicit) assumption that there is a clear-cut separation of scales between the mean and turbulent flows such that $\tau/T \ll 1$ and l/L, where T and L are the characteristic time and length scales of the mean flow and τ and l are the integral time and length scales of turbulence. Of course, as discussed earlier in this paper, this constitutes an oversimplification since in canopy flows, τ/T and l/L can be (and, frequently are) of $O(1)$.

where $\sigma_\epsilon = 1.3$, $C_{\epsilon 1} = 1.44$ and $C_{\epsilon 2} = 1.92$ are empirical (closure) constants. The ϵ-equation here essentially retains the same form as the usual model equation for ϵ commonly utilized in the standard k-ϵ model (Launder and Spalding, 1974). The only difference here is that the drag force effect on the turbulence (embodied in the term F) has been included with the production term P in Equation (45). The ϵ production term is modelled as $C_{\epsilon 1} P/\tau$ [P is determined through Equation (42) on substitution of Equation (23)] in the relaxation time approximation. Finally, the destruction of dissipation term has a simple physical interpretation in terms of a relaxation time; namely, $C_{\epsilon 2}\epsilon^2/\overline{\kappa} = C_{\epsilon 2}\epsilon/\tau$, with τ in this context identified as the lag in the dissipation ϵ, corresponding to the time it will take energy injected (produced) at the energy containing wavenumbers to be reduced to the size of the dissipative wavenumbers.

In Equation (45), we have grouped F with P in the ϵ-equation. In other words, we sensitize the ϵ-equation to the effects of form and viscous drag of the obstacle elements by replacing P with $P + F$ in the 'production of dissipation' term (usually, the effect of the obstacle elements is to enhance the dissipation in the canopy airspace). This treatment is similar to the rationale used by Ince and Launder (1989) for dealing with buoyancy effects on turbulence in buoyancy-driven flows. In these types of flows, the gravitational production term $G \equiv -\beta g_i \overline{u_i' T'}$ (g_i is the gravitational acceleration vector, β is the thermal expansion coefficient, and T' is the virtual temperature fluctuation) is included with P in the transport equation for the viscous dissipation rate. In this regard, our proposed approach for treating F in the ϵ-equation differs from that suggested by Getachew et al. (2000).[14]

6. Whither Wake Production

The closure of the spatially-averaged RANS equation [cf. Equations (12) and (13)] requires a transport equation for $\langle k \rangle \equiv \frac{1}{2}\langle \overline{u_i' u_i'} \rangle$ (i.e., the spatially-averaged turbulence kinetic energy), whereas that for the time-averaged spatially-averaged NS equation [cf. Equations (20) and (21)] requires a transport equation for $\overline{\kappa} \equiv \frac{1}{2}\overline{\langle u_i' \rangle \langle u_i' \rangle}$ (i.e., the time-averaged resolved-scale kinetic energy of turbulence). The model transport equation for $\overline{\kappa}$ in Equation (44) included a source/sink term F whose form

[14] In general, the dissipation rate model (or, equivalently, other forms of the scale-determining equation) is the weakest link in turbulence modelling of complex flows, whether it be for two-equation turbulence models or for Reynolds stress transport models. Although more sophisticated dissipation models have been developed, they seem to lack the general applicability of the simple model used here.

can be systematically derived in terms of the drag force term that appears in the mean momentum equation [cf. Equations (38) and (39)].

The budget equation for $\langle k \rangle$ can be derived by applying the spatial averaging operator to the standard transport equation for k to give

$$\begin{aligned}\frac{\partial \langle k \rangle}{\partial t} + \langle \bar{u}_j \rangle \frac{\partial \langle k \rangle}{\partial x_j} \;=\; & -\langle \overline{u_i' u_j'} \rangle \frac{\partial \bar{u}_i}{\partial x_j} \\ & - \frac{\partial}{\partial x_j} \left[\frac{1}{2} \langle \overline{u_j' u_i' u_i'} \rangle + \langle \overline{u_j' p'} \rangle + \underbrace{\frac{1}{2} \langle \bar{u}_j'' \overline{u_i' u_i'} \rangle}_{\mathrm{I}} \right] \\ & + \nu \frac{\partial^2 \langle k \rangle}{\partial x_j^2} - \varepsilon - \underbrace{\left\langle \overline{u_i' u_j'}'' \frac{\partial \bar{u}_i''}{\partial x_j} \right\rangle}_{\mathrm{II}} . \end{aligned} \tag{46}$$

In Equation (46), $\varepsilon \equiv \nu \langle \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} \rangle$ is the isotropic turbulent dissipation rate for $\langle k \rangle$.

The transport equation for $\langle k \rangle$ contains two additional terms (designated I and II) that need to be approximated. Term I corresponds to the dispersive transport of $\langle k \rangle$ [analogous to the dispersive flux of momentum in Equations (12) and (13)]. Term II can be identified as a wake production term (see Raupach and Shaw, 1982) which accounts for the conversion of mean kinetic energy to turbulent energy in the obstacle wakes by working of the mean flow against the drag. This term is analogous to the F term that appears in the transport equation for $\overline{\kappa}$, but unlike F whose form can be systematically derived from the form and viscous drag force term that appears in the mean momentum equation, the link (if any) between the wake production term in Equation (46) and the drag force term in the mean momentum equation is less obvious. For example, Raupach and Shaw (1982) showed that provided (1) the dispersive stress $\langle \bar{u}_i'' \bar{u}_j'' \rangle$ and the dispersive transport of $\langle k \rangle$ are both negligible and (2) the mean kinetic energy is not directly dissipated to heat in the canopy, the wake production term can be approximated as follows:

$$-\langle (\overline{u_i' u_j'})'' \frac{\partial \bar{u}_i''}{\partial x_j} \rangle = -\langle \bar{u}_i \rangle \bar{f}_i = 2 C_D \hat{A} Q \underbrace{\left(\frac{1}{2} \langle \bar{u}_i \rangle \langle \bar{u}_i \rangle \right)}_{K}, \tag{47}$$

where K represents the mean kinetic energy (kinetic energy of the spatially-averaged time-mean flow). Note that use of Equation (47) as a model for the wake production term strictly provides a *source* term in the transport equation for $\langle k \rangle$ and, physically, corresponds to the conversion of mean kinetic energy (MKE) to turbulence kinetic energy (TKE) [MKE $\to$ TKE which here is equated simply to the work done by the flow against form drag $-\langle \bar{u}_i \rangle \bar{f}_i$].

Generally, the inclusion of the wake production term of Equation (47) in the transport equation for $\langle k \rangle$ has been found to result in an overestimation of the turbulence level within a vegetative canopy. For example, Wilson and Shaw (1977) included a wake production term in their streamwise normal stress equation and found that their model overestimated the streamwise and vertical turbulence intensities in a corn canopy despite the fact that they tuned their mixing length. Furthermore, Wilson (1985) found that a zone of reduced TKE ("quiet zone") in the near lee of a shelter belt cannot be reproduced correctly with the inclusion of a wake production term in the normal stress transport equations[15] but, rather, required the exclusion of this source term coupled with the inclusion of a sink term corresponding approximately to Equation (31). Likewise, Green (1992) and Liu et al. (1996) found that it was necessary to include also a sink term in the budget equation for $\langle k \rangle$ and, to this purpose, modelled the "wake" production term in the budget equation for $\langle k \rangle$ in an *ad hoc* manner as

$$-\langle (\overline{u_i' u_j'})'' \frac{\partial \bar{u}_i''}{\partial x_j} \rangle = \beta_P C_D \hat{A} Q \underbrace{\left(\frac{1}{2} \langle \bar{u}_i \rangle \langle \bar{u}_i \rangle \right)}_{K} - \beta_d C_D \hat{A} Q \underbrace{\left(\frac{1}{2} \langle \overline{u_i' u_i'} \rangle \right)}_{\langle k \rangle}, \quad (48)$$

which includes a gain to $\langle k \rangle$ from conversion of the mean kinetic energy K to turbulence energy at the larger scales (source term) and a loss from $\langle k \rangle$ of the large-scale turbulence energy to smaller (wake) scales (sink term). In Equation (48), β_P and β_d are $O(1)$ empirical dimensionless constants with no particular physical significance. More specifically, Green (1992) argued heuristically that the sink term in Equation (48) was required to account for the accelerated cascade of $\langle k \rangle$ from large to small scales due to the presence of the roughness elements (arising from the rapid dissipation of fine-scale wake eddies in a plant canopy). Liu et al. (1996) in their $\langle k \rangle$ equation used $\beta_P = 2$ and $\beta_d = 4$, generally giving a model where the overall effect of the "source" term in Equation (48) is to act as a sink term within the canopy. Indeed, Liu et al. (1996) noted that the *ad hoc* inclusion of the sink term (second term on the right-hand side) of Equation (48) [which was inserted in hindsight] was important, for otherwise they found that their predicted

[15] The inclusion of a wake production (source) term generally led to an increase of the TKE level in the near lee of a porous barrier, contrary to the available experimental observations.

$\langle k \rangle$ was "about 100% larger than the experimental measurements when the second term was ignored".

Sanz (2003) described an alternative method for the determination of the source model coefficients β_P and β_d in Equation (48). Sanz recognized the confusion that had reigned for some time as to the 'correct' source terms in TKE and dissipation-rate equations for canopy flow. His solution was to constrain particular pre-existing (and heuristic) expressions for the forms of the sinks S_k and S_ϵ in the model equations for TKE ("k") and its dissipation rate ϵ, though without identifying k specifically with $\langle k \rangle$ or with $\overline{\kappa}$ (i.e., a precise interpretation of k was not provided). The constraints emerged by requiring that the k-ϵ model (with these sources) should reproduce an exponential mean wind profile in the canopy layer, in the special case of a horizontally-uniform, neutral plant canopy flow in which both leaf area density and the effective turbulence lengthscale $k^{3/2}/\epsilon$ were (by requirement) height independent. Evidently, this approach does not offer the generality provided here.

In contrast, the additional source/sink term F that appears in the budget equation for $\overline{\kappa}$ can be systematically derived from rate of working of the turbulent velocity fluctuations against the fluctuating drag force, and appears naturally in the derivation of the budget equation for $\overline{\kappa}$. The form of the source/sink term F results from a series expansion for f_i that is truncated systematically at second order in the velocity fluctuations $\langle u_i' \rangle$. All constants in this model for F are derived explicitly from the expansion procedure; no adjustable constants arise and no additional *ad hoc* modifications are applied to F, in contrast to the mentioned treatments of the transport equation for $\langle k \rangle$ [cf. Equation (48)]. Even though the 'turbulence kinetic energy' $\overline{\kappa} \equiv \frac{1}{2}\overline{\langle u_i' \rangle \langle u_i' \rangle}$ (time-averaged, resolved-scale kinetic energy of turbulence) used in our turbulence closure model is different from the usual form of the spatially-averaged turbulence kinetic energy $\langle k \rangle \equiv \frac{1}{2}\langle \overline{u_i' u_i'} \rangle$, they are nevertheless related as follows:

$$\langle k \rangle - \overline{\kappa} = \frac{1}{2}\left(\langle \overline{u_i'' u_i''} \rangle - \langle \bar{u}_i'' \bar{u}_i'' \rangle \right) = \frac{1}{2}\langle \overline{(u_i')''(u_i')''} \rangle. \tag{49}$$

Note that the difference between $\langle k \rangle$ and $\overline{\kappa}$ is proportional to the difference between the two forms of dispersive stress that appear in the spatially-averaged RANS equation and the time-averaged, spatially-averaged NS equation. However, note that this difference can be expressed as the spatial average of time averages of the departures of velocity fluctuations from their spatial (volume) average. Since this term involves a "perturbation of a perturbation", it seems reasonable to *assume*[16] that

$$0 \approx \frac{1}{2}\left|\, \langle \overline{(u_i')''(u_i')''} \,\rangle \,\right| \ll \max(\, \langle k \rangle, \overline{\kappa}\,). \tag{50}$$

With this assumption, $\langle k \rangle$ and $\overline{\kappa}$ are expected to be almost equal in value (viz., $\langle k \rangle \approx \overline{\kappa}$). This, together with Equation (22), implies that

$$\langle \bar{u}_i'' \bar{u}_j'' \rangle \approx \langle \overline{u_i'' u_j''} \rangle, \tag{51}$$

or, in other words, the dispersive stresses are expected to be approximately equal to the spatial average of the high-frequency turbulent stresses. Finally, with reference to Equation (22), this also implies that $\overline{\langle u_i' \rangle \langle u_j' \rangle} \approx \langle\, \overline{u_i' u_j'}\, \rangle$. The latter approximation will be used in III to compare model predictions of $\overline{\langle u_i' \rangle \langle u_j' \rangle}$ with the diagnosed values of $\langle\, \overline{u_i' u_j'}\, \rangle$ obtained from a high-resolution RANS simulation.

Interestingly, the 'zeroth-order' term in our expansion of F in Equation (38), rewritten below

$$F \equiv \overline{\langle u_i' \rangle f_i'} = -2 C_D \hat{A} Q \underbrace{\left(\frac{1}{2} \overline{\langle u_i' \rangle \langle u_i' \rangle} \right)}_{\overline{\kappa}} + \text{H.O.T}, \tag{52}$$

where H.O.T denotes higher-order correction terms, is analogous to the sink term contribution for the wake production term in the Liu et al. (1996) model of Equation (48) [except the factor here is 2, rather than $\beta_d = 4$]. Note that the higher-order correction terms for F can have either sign implying that they are source/sink terms. More importantly, it needs to be emphasized that the leading-order term of F is a *sink* term, and not a source term implying that in the transport equation for $\overline{\kappa}$ the conversion of MKE to TKE is *ipso facto* absent.[17] In this sense, the transport equation derived here for $\overline{\kappa}$ is reminiscent of the transport equation for shear kinetic energy (SKE) originally proposed by Wilson (1988). Wilson formulated a heuristic approach in which

[16] No experimental observations are available to support or refute this assumption. It is conceivable that large-eddy simulations (LES) of plant or urban canopy flows may provide "data" that can be used to evaluate the validity of this assumption.

[17] Although Wang and Takle (1995b) use the transport equation for $\overline{\kappa}$ for their simulations of flows near shelter belts, they seem to have incorporated wrongly a source term $S_{\text{MKE}} \equiv C_D \hat{A} \big(\langle \bar{u}_i \rangle \langle \bar{u}_i \rangle \big)^{3/2}$ in the $\overline{\kappa}$-equation representing MKE conversion to $\overline{\kappa}$ by drag of the shelter belt on the flow. Equation (52) shows that the inclusion of the MKE conversion term in the $\overline{\kappa}$-equation is not self-consistent with the momentum sink terms introduced into the transport equations for the time-averaged spatially-averaged mean wind. Furthermore, Wilson and Mooney (1997) reported that using the source term S_{MKE} in the $\overline{\kappa}$ transport equation resulted in a drastic overestimation of peak TKE levels near the porous barrier.

he partitioned the turbulence energy into two spectral bands; namely, bands of turbulent motion corresponding to the large-scale or shear kinetic energy and to small-scale or wake kinetic energy (WKE). In this approach, the MKE conversion due to the drag forces was assumed to be re-deposited as fine-scale WKE [viz., in small eddies on the scale of the canopy elements (twigs, leaves, etc.) in the fine-scaled plant canopy] where it was rapidly dissipated as heat. The conversion of SKE to WKE was modelled here simply as the sum of two terms, the first representing the conventional dissipation due to the vortex-stretching mechanism and the second arising from the rate of working of turbulence against the vegetation drag resulting in an additional sink term in the SKE transport equation of the form

$$\epsilon_{\rm fd} = 2C_D\hat{A}\langle\bar{u}\rangle\Big(\langle\overline{u'^2}\rangle + \frac{1}{2}\langle\overline{v'^2}\rangle + \frac{1}{2}\langle\overline{w'^2}\rangle\Big). \tag{53}$$

Note that the sink term $\epsilon_{\rm fd}$ appearing in the SKE transport equation is very similar to the leading order term of F (sink term) appearing in the $\overline{\kappa}$ transport equation [cf. Equation (52) with Equation (53)], the *mathematical* difference being solely due to Wilson's approximation that $u\sqrt{u^2+v^2+w^2} \approx u^2$ (etc.). However the mathematical similarity should not cause one to lose sight of the important logical difference, for Wilson's instantaneous velocity (u, v, w) lacked the proper designation as a spatial average.

Despite the outward similarities between the current treatment of $\overline{\kappa}$ and Wilson's treatment of SKE, it is legitimate to ask whether $\overline{\kappa}$ is exactly coincident with SKE defined by Wilson. The implicit assumption in Wilson's (1988) two-band "spectral division" for the turbulence energy resides in the presumption that there is a clear cut separation between the large (SKE) and small (WKE) scale components of the turbulence and that MKE and large-scale TKE lost due to the action of form drag must necessarily be re-deposited as fine-scale, rapidly dissipated eddies in the WKE component. In this two-band spectral decomposition approach, the characteristic length scale separating the large- and small-scale components of turbulence is a physical scale imposed by the elements of the fine-scaled plant canopy itself (viz., direct interaction of the flow with the foliage imposes a new, smaller length scale on the flow corresponding to the leaf or twig scale, and it is this range of scales which is associated with the fine-scale WKE). While the decomposition of the turbulence into a large-scale SKE and fine-scale WKE is reasonable for a fine-grained (and dense) plant canopy, it appears to be less appropriate for a coarse-grained urban canopy where the wake scales of motion behind buildings (element wakes) are not small, but rather frequently comparable to the scales associated with the shear production of turbulence energy.

We note that $\overline{\kappa} \equiv \frac{1}{2}\overline{\langle u_i' \rangle \langle u_i' \rangle}$ (time-mean locally-spatially-filtered turbulence kinetic energy) represents a restricted spectrum of turbulent motions and, in this sense, is similar to SKE defined by Wilson (1988). However, the basic difference in $\overline{\kappa}$ and SKE arises from the fact that the implicit scale separating "large" and "small" scales of turbulent motion in $\overline{\kappa}$ is obtained from the application of an explicit spatial filtering (volume-averaging) on the turbulent velocity fluctuations u_i' itself [in contrast to SKE where the separation between large and small scales of motion devolves from a physical length scale imposed on the flow by the canopy elements (e.g., fine-grained foliage in a plant canopy)]. Furthermore, the scale separating large and small turbulent motions in the two-band spectral representation of Wilson (1988) is not explicitly specified, other than stating it is to guarantee that the SKE excludes the TKE residing in the fine wake scales. In contrast, the scale separating large and small turbulent motions in the definition of $\overline{\kappa}$ is explicitly specified by the filter width Δ that defines the spatial filter applied to the turbulent velocity fluctuations [cf. Equations (1) and (4)].

To explore this concept further, it is informative to write $\overline{\kappa}$ in terms of the following spectral representation:

$$\overline{\kappa} \equiv \frac{1}{2}\overline{\langle u_i' \rangle \langle u_i' \rangle} = \frac{1}{2}\int_{-\infty}^{\infty} \left|\widehat{G}(\mathbf{k})\right|^2 \Phi(\mathbf{k}) d\mathbf{k}, \tag{54}$$

where $\mathbf{k}$ is the wavenumber vector and $\Phi(\mathbf{k})$ is the turbulence energy spectral density defined over the *entire spectrum of turbulent motions*. In Equation (54), $\widehat{G}(\mathbf{k})$ is the Fourier transform of the top-hat filter exhibited in Equation (4) given by

$$\widehat{G}(\mathbf{k}) = \prod_{i=1}^{3} \frac{\sin(\Delta_{(i)}\mathrm{k}_{(i)}/2)}{\Delta_{(i)}\mathrm{k}_{(i)}/2}, \tag{55}$$

where k_i is the i-th component of the wavenumber vector $\mathbf{k}$, Δ_i is the filter width in the x_i-direction (with total filter width Δ specified by $\Delta = V^{1/3} = (\Delta_1\Delta_2\Delta_3)^{1/3}$), and the round parentheses around an index i [viz., (i)] indicates that there is no summation over i for this repeated index. Equation (54) shows explicitly that $\overline{\kappa}$ only incorporates the turbulence energy in a low wavenumber band with wavenumber contributions confined largely to the range $|\mathbf{k}| \lesssim \pi/\Delta$ determined by the transfer function $\left|\widehat{G}(\mathbf{k})\right|^2$ of the spatial filter defined in Equation (4). This spatial filter effectively removes (attenuates) small-scale flow features that are less than an externally introduced (imposed) length scale Δ (filter width). However, we emphasize that this spatial filtering operation on the velocity field does not introduce a clear-cut separation between resolved and sub-filter scales because (1) the top-hat filter allows a frequency overlap between the resolved and sub-filter scales and (2) the turbulence energy spectrum is continuous so that the smallest resolved scales of motion are close to those of the largest sub-filter scales of motion.

The time-average of the spatially-averaged NS equations leads logically to the consideration of the transport equation for $\overline{\kappa}$, rather than $\langle k \rangle$. This averaging scheme results naturally in a two-band spectral representation for the TKE that is similar to that proposed by Wilson (1988). However, while the dual-band model of Wilson partitions the turbulence energy into a large-scale SKE and fine-scale WKE with the length scale separating these two bands determined by the scale of the individual canopy elements (e.g., leaf and twig scale in a plant canopy), the dual-band model here is perhaps interpreted best as partitioning the turbulence energy into a resolved scale and sub-filter scale band with the separation between these two scales externally imposed by the filter width Δ of the spatial filter applied in the averaging operation. In this interpretation, $\overline{\kappa}$ should be identified with the resolved turbulence kinetic energy ($\mathrm{TKE}^{[>]}$) that is due to energetic motions in the flow at scales *greater* than the externally determined length scale Δ (filter width). A key aspect of the modelled transport equation for $\overline{\kappa} \equiv \mathrm{TKE}^{[>]}$ [cf. Equation (44)] is the exchange or conversion of kinetic energy between $\mathrm{TKE}^{[>]}$ and the kinetic energy of turbulent motions at scales less than Δ ($\mathrm{TKE}^{[<]}$). The leading-order term in F is negative, implying a "forward scatter" of $\mathrm{TKE}^{[>]}$ into $\mathrm{TKE}^{[<]}$, a physical mechanism which reduces the level of $\mathrm{TKE}^{[>]}$. More significantly, the leading-order term of F does not involve a source term related to the conversion of MKE to $\mathrm{TKE}^{[>]}$. However, higher-order terms of F may possibly embody an increase of the level of $\mathrm{TKE}^{[>]}$ due to interactions between the time-mean spatially-averaged mean flow and the resolved-scale turbulence.[18]

Applying a spatial filter to the equations of motion results in a reduction in the complexity and information content of the flow, but introduces a length scale into the description of the fluid dynamics, namely the width Δ of the filter used. Assuming that the filter width applied is within the inertial range of scales of fluid motion, the application of the spatial filter will cause the turbulence kinetic energy wavenumber

[18] The sign of many of the higher-order terms in F is indefinite, so some of these interactions between the mean flow and the resolved-scale turbulence (viz., turbulent motions with length scales larger than the filter width Δ, or equivalently with wavenumbers $|\mathbf{k}| \lesssim \pi/\Delta$) may be sink terms that contribute to the reduction of $\mathrm{TKE}^{[>]}$.

spectrum in three dimensions of the filtered turbulent motions to roll-off rapidly below the length scale Δ as $|\mathbf{k}|^{-2-5/3}$ [cf. Equations (54) and (55)], instead of continuing at the slower Kolmogorov scaling law, $|\mathbf{k}|^{-5/3}$. This roll off shortens the inertial range for the spatially-filtered velocity field, resulting in a reduction in the energy associated with the higher wavenumbers in the filtered velocity and an increased dissipation in the range of scales $|\mathbf{k}|\Delta \gg \pi$, implying an increased drain of the energy of the resolved (large) scales by those of the sub-filter (small) scales. The effects of the small-scale fluid motions on the resolved energetics of turbulence in the canopy flow (embodied in $\overline{\kappa}$) must physically be described by an increased (effective) "viscous" dissipation ϵ_{eff} of the resolved scales of motion that now becomes effective at $|\mathbf{k}|\Delta \approx \pi$ (rather than at the larger wavenumber associated with the Kolmogorov microscale), with ϵ_{eff} consisting of two basic contributions: namely, the conventional free-air dissipation associated with the spectral eddy cascade and an additional dissipation ($\approx 2C_D\hat{A}Q\overline{\kappa}$) devolving from air/obstacle interactions that result in additional source/sink terms in the $\overline{\kappa}$-equation after a spatial filtering operation has been explicitly applied to the instantaneous velocity fluctuations. The effect of this additional dissipation is embodied in the leading-order term of F [cf. Equation (52)].

7. Conclusions

In this paper, we showed how a modified k-ϵ model for the prediction of the time-mean spatially averaged wind and turbulence fields in a canopy can be derived 'sytematically' (or, at least in a self-consistent manner) by time-averaging the spatially-averaged NS equation. This procedure ensures the mathematical and logical consistency of parameterization of source/sink terms in the mean momentum equations and in the supporting transport equations for $\overline{\kappa}$ and ϵ. Given a parameterization for the momentum source/sink f_i in the spatially-averaged NS equation that represents the effects of the form and viscous drag in the urban canopy on the flow, a series expansion is applied to this term with a truncation at the second order in the velocity fluctuations $\langle u_i' \rangle$. Truncation at this order produces a quadratically nonlinear model for the time-averaged momentum sink $\bar{f}_i$, and also permits the corresponding source/sink term in the transport equation for $\overline{\kappa}$ to be methodically obtained.

The current approach of time-averaging the spatially-averaged NS equations is logically (and self-consistently) linked with the transport equation for $\overline{\kappa}$ (time-averaged, resolved-scale kinetic energy of turbulence), rather than that for $\langle k \rangle$ (spatially-averaged turbulence kinetic energy). Interestingly, $\overline{\kappa}$ very explicitly and naturally does not include TKE at scales smaller than the filter width $\Delta = V^{1/3}$. In this sense, $\overline{\kappa}$ here is similar to SKE considered by Wilson (1988). The basic difference in $\overline{\kappa}$ and SKE arises from the fact that the small length scale imposed on the flow by the individual canopy elements (e.g., individual leaves) is the characteristic scale that is used (implicitly) in Wilson's approach to distinguish the large scales associated with SKE with the small scales associated with WKE, whereas in the present approach an externally imposed filter width Δ is used to distinguish between the resolved scales associated with $\overline{\kappa}$ and the kinetic energy associated with the sub-filter (unresolved) scales. The current approach seems more applicable to coarse-scaled cuboid arrays where the wake scales of the individual canopy elements are not necessarily small in comparison to the scales associated with the shear production. With this distinction, the present approach can be interpreted as providing a formal basis for Wilson's dual-band spectral decomposition of the TKE [viz., we have provided a precise mathematical formulation of the heuristic ideas for "spectral division" of the TKE sketched earlier by Wilson (1988) for parameterizing turbulence in RANS models which treat the interaction of the wind with obstacles using a distributed momentum sink in the mean momentum equations].

Appendix A: Dispersive Stress Tensor Model

While there currently exists no reference data to guide the modelling of the 'conventional' dispersive stress tensor $\langle \bar{u}_i'' \bar{u}_j'' \rangle$, it is nevertheless possible to construct a structural model for this quantity as follows. To begin, consider the Reynolds equation for mean momentum conservation in a neutrally buoyant flow that obtains at any point in the canopy airspace:

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{u}_i)}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j} - \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} = 0. \tag{56}$$

After applying the volume-averaging operator of Equation (5) to Equation (56), the dispersive stress $\langle \bar{u}_i'' \bar{u}_j'' \rangle$ arises from the noncommutation of the product operator and the volume-averaging operator in the nonlinear convective term. In view of this, to construct a structural model for the dispersive stress we consider a simplification of the nonlinear convective term of Equation (56) that is obtained by volume-averaging (smoothing) the *advection* velocity; viz., replacing the nonlinear convective term in Equation (56) by $\langle \bar{u}_j \rangle \partial \bar{u}_i / \partial x_j$ to give

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\langle \bar{u}_j \rangle \bar{u}_i)}{\partial x_j} + \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \overline{u_i' u_j'}}{\partial x_j} - \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} = 0. \tag{57}$$

In Equation (57), the time-mean velocity $\bar{u}_i$ is advected by a smoothed velocity $\langle \bar{u}_j \rangle$ obtained by volume averaging. Replacing the advection velocity $\bar{u}_j$ by $\langle \bar{u}_j \rangle$ simplifies the problem by reducing the nonlinear effects of the convective term.

The spatially filtered time-mean fluid velocity $\langle \bar{u}_i \rangle$ is related to the time-mean fluid velocity $\bar{u}_i$ as $\langle \bar{u}_i \rangle = G \star \bar{u}_i$, where G is an operator whose Green's function is the filter defined by Equations (1) and (4). The "inverse" is denoted by $\bar{u}_i = G^{-1} \star \langle \bar{u}_i \rangle$, assuming that a formal inverse G^{-1} of G exists. Using this terminology, Equation (57) [simplified RANS equation with the degree of nonlinearity of the convective term reduced] can be written more informatively as follows:

$$\begin{aligned} \frac{\partial (G^{-1} \star \langle \bar{u}_i \rangle)}{\partial t} &+ \frac{\partial (\langle \bar{u}_j \rangle \bar{u}_i)}{\partial x_j} + \frac{\partial (G^{-1} \star \langle \bar{p} \rangle)}{\partial x_i} \\ &+ \frac{\partial (G^{-1} \star \langle \overline{u_i' u_j'} \rangle)}{\partial x_j} - \nu \frac{\partial^2 (G^{-1} \star \langle \bar{u}_i \rangle)}{\partial x_j^2} = 0. \end{aligned} \tag{58}$$

Applying the spatially-averaging operator $G\star$ to Equation (58) gives explicitly

$$\begin{aligned} \frac{\partial \langle \bar{u}_i \rangle}{\partial t} &+ \frac{\partial (\langle \bar{u}_j \rangle \langle \bar{u}_i \rangle)}{\partial x_j} + \frac{\partial \langle \bar{p} \rangle}{\partial x_i} + \frac{\partial \langle \overline{u_i' u_j'} \rangle}{\partial x_j} - \nu \frac{\partial^2 \langle \bar{u}_i \rangle}{\partial x_j^2} = \\ & -\Bigg\{ \left[G\star, \frac{\partial}{\partial t} \right] \bar{u}_i + \frac{\partial}{\partial x_j} \Big(\langle \bar{u}_i \langle \bar{u}_j \rangle \rangle - \langle \bar{u}_i \rangle \langle \bar{u}_j \rangle \Big) \\ & + \left[G\star, \frac{\partial}{\partial x_j} \right] P(\bar{u}_i, \langle \bar{u}_j \rangle) + \left[G\star, \frac{\partial}{\partial x_i} \right] \bar{p} \\ & + \left[G\star, \frac{\partial}{\partial x_j} \right] (\overline{u_i' u_j'}) - \left[G\star, \frac{\partial}{\partial x_j} \right] \left(\nu \frac{\partial \bar{u}_i}{\partial x_j} \right) \Bigg\}, \end{aligned} \tag{59}$$

where $P(f,g) \equiv fg$ is the product operator.

Now, in view of the spatial averaging theorem of Equation (2) used with G specified by Equation (4), most of the terms on the right-hand side of Equation (59) vanish on application of the no-slip and impermeability conditions on the velocity field at the surfaces of the obstacles inside the averaging volume V. The non-vanishing terms involving the commutator brackets on the right-hand side of Equation (59) are $-[G\star, \partial/\partial x_i]\bar{p}$ and $[G\star, \partial/\partial x_j](\nu\partial\bar{u}_i/\partial x_j)$ which in view of Equation (14) can be identified physically with, respectively, $\bar{f}_{Fi}$ and $\bar{f}_{Vi}$ (namely, the form and viscous drag force vectors exerted on a unit mass of air in the averaging volume). Noting that $\bar{f}_i = \bar{f}_{Fi} + \bar{f}_{Vi}$, Equation (59) simplifies to the following form

$$\frac{\partial\langle\bar{u}_i\rangle}{\partial t} + \frac{\partial\big(\langle\bar{u}_j\rangle\langle\bar{u}_i\rangle\big)}{\partial x_j} + \frac{\partial\langle\bar{p}\rangle}{\partial x_i} + \frac{\partial\langle\overline{u'_i u'_j}\rangle}{\partial x_j} - \nu\frac{\partial^2\langle\bar{u}_i\rangle}{\partial x_j^2} = -\frac{\partial}{\partial x_j}\Big(\langle\bar{u}_i\langle\bar{u}_j\rangle\rangle - \langle\bar{u}_i\rangle\langle\bar{u}_j\rangle\Big) + \bar{f}_i, \tag{60}$$

which on comparison with Equations (12) and (13) implies the following structural model for the dispersive stress tensor expressed in terms of the spatially-averaged time-mean velocity $\langle\bar{u}_i\rangle$:

$$\begin{aligned}\langle\bar{u}''_i\bar{u}''_j\rangle &= \langle\bar{u}_i\langle\bar{u}_j\rangle\rangle - \langle\bar{u}_i\rangle\langle\bar{u}_j\rangle \\ &= \Big\langle\big(G^{-1}\star\langle\bar{u}_i\rangle\big)\langle\bar{u}_j\rangle\Big\rangle - \langle\bar{u}_i\rangle\langle\bar{u}_j\rangle.\end{aligned} \tag{61}$$

Note that the model for $\langle\bar{u}''_i\bar{u}''_j\rangle$ in Equation (61) is derived directly from a regularized form of the RANS equation, and as such, is consistent with various transformation symmetries (e.g., Galilean invariance, time invariance, etc.) of this equation. Furthermore, the model for $\langle\bar{u}''_i\bar{u}''_j\rangle$ does not involve any model (closure) coefficients and suggests that the implied dispersive stress model must necessarily involve an explicit filtering and an inversion operation. Finally, in light of Equations (49), (50), and (51), the structural model for $\langle\bar{u}''_i\bar{u}''_j\rangle$ can also be used as a model for $\langle\overline{u''_i u''_j}\rangle$.

To implement the model for $\langle\bar{u}''_i\bar{u}''_j\rangle$, we are required to construct an approximation for the inverse operator G^{-1} (which in view of Equation (4) is the inverse operator for the volume-averaging operation). To proceed with this construction, consider the top-hat filter of Equation (4) applied in a single direction (say, the x-direction) only. Now perform a Taylor series expansion of a flow quantity $\phi(y)$ about the fixed point x to give

$$\phi(y) = \sum_{n=0}^{\infty}\frac{1}{n!}(y-x)^n\frac{\partial^n\phi(x)}{\partial x^n} \tag{62}$$

and insert this expansion into the one-dimensional version of Equation (1) to give

$$\langle\phi\rangle(x) = \sum_{n=0}^{\infty}\frac{\alpha^{(n)}}{n!}\frac{\partial^n\phi(x)}{\partial x^n} \tag{63}$$

where $\alpha^{(n)}$ denotes the n-th order moment of the convolution kernel G:

$$\alpha^{(n)} \equiv (-1)^n \int_{-\infty}^{\infty} s^n G(s)\, ds. \tag{64}$$

For a symmetric filter considered here, all odd moments vanish (viz., $\alpha^{(n)} = 0$ for n odd). Equation (63) shows that we can approximate the spatial filtering operation by truncating the Taylor series expansion to $(N+1)$ terms to give a filtering operation that is defined by a low-order differential filter:

$$\langle \phi \rangle(x) \approx \sum_{n=0}^{N} \frac{\alpha^{(n)}}{n!} \frac{\partial^n \phi(x)}{\partial x^n}. \tag{65}$$

The unfiltered flow quantity ϕ can be expressed formally as

$$\phi(x) \approx \left(\sum_{n=0}^{N} \frac{\alpha^{(n)}}{n!} \frac{\partial^n}{\partial x^n} \right)^{-1} \langle \phi(x) \rangle. \tag{66}$$

A simple explicit approximation of the inverse operator for $N = 2$ (valid to second order in the filter width Δ_x) can be written as [using a formal Taylor series expansion for the inverse operator of Equation (66) valid to second order]

$$\phi(x) \approx \left(1 - \frac{\Delta_x^2}{24} \frac{\partial^2}{\partial x^2} \right) \langle \phi(x) \rangle, \tag{67}$$

where 1 is the identity operator. In Equation (67), we have used the fact that $\alpha^{(2)} = \Delta_x^2/12$ for a one-dimensional top-hat filter with filter width Δ_x. The inversion operator in three-dimensions can be obtained from composing three one-dimensional filters to give approximately

$$\begin{aligned} \phi(\mathbf{x}) &\approx \left(1 - \frac{\Delta_x^2}{24} \frac{\partial^2}{\partial x^2} - \frac{\Delta_y^2}{24} \frac{\partial^2}{\partial y^2} - \frac{\Delta_z^2}{24} \frac{\partial^2}{\partial z^2} \right) \langle \phi(\mathbf{x}) \rangle \\ &\equiv H_\Delta(\langle \phi(\mathbf{x}) \rangle), \end{aligned} \tag{68}$$

where H_Δ denotes the Helmholtz operator with $\Delta \equiv (\Delta_x \Delta_y \Delta_z)^{1/3}$ being the effective filter width in three dimensions.

Appendix B: Non-perturbative Evaluation of Source Terms

The calculation of $\bar{f}_i$ and F described in Sections 4 and 5 was based on expanding the spatially filtered total instantaneous velocity amplitude $(\langle u_j \rangle \langle u_j \rangle)^{1/2}$ using the binomial series and time averaging the various terms in this expansion. However, the convergence of this series required that the time-averaged resolved scale kinetic energy of turbulence be sufficiently small compared to the magnitude of the spatially-averaged time-mean wind speed such that the condition for absolute convergence of the binomial series (see footnote 10) is satisfied. It is conceivable that this condition for absolute convergence of the binomial series is violated in a canopy flow for cases of strong turbulence where $\langle u_j'\rangle\langle u_j'\rangle \gtrsim \langle\bar{u}_j\rangle\langle\bar{u}_j\rangle$. This appendix offers an alternative method (based on a non-perturbative approach) for the calculation of $\bar{f}_i$ and F that is valid for the case of strong turbulence (and, also for weak turbulence of course). The actual calculation method for the non-perturbative evaluation of $\bar{f}_i$ and F is quite complex, but the methodology is generally applicable and does not require the existence of a small expansion parameter $\delta \equiv |\langle u_i'\rangle|/Q$ [cf. Equation (30)].

To begin, recall that

$$\begin{aligned}\bar{f}_i &\equiv -C_D\hat{A}\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i\rangle} \\ &= -C_D\hat{A}\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\left(\langle\bar{u}_i\rangle + \langle u_i'\rangle\right)}\end{aligned} \tag{69}$$

and

$$\begin{aligned}F &\equiv \overline{\langle u_i'\rangle f_i'} \\ &= -C_D\hat{A}\left(\langle\bar{u}_i\rangle\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i'\rangle} + \overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i'\rangle\langle u_i'\rangle}\right).\end{aligned} \tag{70}$$

The computation of $\bar{f}_i$ and F requires the consideration of three ensemble (or probability) averages; namely,

$$\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}} = \iiint_{-\infty}^{\infty}\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\Psi\Big(\langle\mathbf{u}'\rangle\Big)\,d\langle\mathbf{u}'\rangle; \tag{71}$$

$$\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i'\rangle} = \iiint_{-\infty}^{\infty}\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i'\rangle\Psi\Big(\langle\mathbf{u}'\rangle\Big)\,d\langle\mathbf{u}'\rangle; \tag{72}$$

and

$$\begin{aligned}\overline{\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2}\langle u_i'\rangle\langle u_i'\rangle} \;=\; &\iiint_{-\infty}^{\infty}\left(\langle u_j\rangle\langle u_j\rangle\right)^{1/2} \\ &\times\langle u_i'\rangle\langle u_i'\rangle\Psi\Big(\langle\mathbf{u}'\rangle\Big)\,d\langle\mathbf{u}'\rangle.\end{aligned} \tag{73}$$

Here, $\Psi(\langle\mathbf{u}'\rangle)$ is the joint probability density function (PDF) of the spatially-averaged velocity fluctuations vector $\langle\mathbf{u}'\rangle \equiv \langle u_i'\rangle$. Note that in writing Equations (71) to (73), for simplicity of notation we have used $\langle\mathbf{u}'\rangle$ to denote values in the sample space of the random vector corresponding to the spatially filtered velocity fluctuations; whereas, in Equations (69) and (70) $\langle \mathbf{u}' \rangle \equiv \langle u_i' \rangle$ has been used to denote the random vector itself. The precise useage of $\langle \mathbf{u}' \rangle$ in the material that follows should be clear from the context.

To evaluate the integrals in Equations (71), (72), and (73), first note that the spatial average of the total instantaneous velocity can be decomposed as $\langle u_i \rangle = \langle \bar{u}_i \rangle + \langle u_i' \rangle$. Next, it is convenient to express the spatially-averaged time-mean velocity vector $\langle \bar{\mathbf{u}} \rangle \equiv \langle \bar{u}_i \rangle$ and the spatially-averaged velocity fluctuation vector $\langle \mathbf{u}' \rangle \equiv \langle u_i' \rangle$ in spherical coordinates. To this end, let the spherical coordinates of $\langle \bar{\mathbf{u}} \rangle$ and $\langle \mathbf{u}' \rangle$ be $\big(|\langle \bar{\mathbf{u}} \rangle|, \bar{\theta}, \bar{\phi}\big)$ and $\big(|\langle \mathbf{u}' \rangle|, \theta', \phi'\big)$, respectively. Here, $|\cdot|$ denotes the magnitude (or, Euclidean length) of the vector and θ and ϕ are the co-latitudinal (or, zenith) and azimuthal angles, respectively. Hence, the Cartesian components of $\langle \mathbf{u}' \rangle$ can be expressed in terms of its spherical coordinates $\big(|\langle \mathbf{u}' \rangle|, \theta', \phi'\big)$ as $\langle u_1' \rangle = \big|\langle \mathbf{u}' \rangle\big| \sin\theta' \cos\phi'$, $\langle u_2' \rangle = \big|\langle \mathbf{u}' \rangle\big| \sin\theta' \sin\phi'$, and $\langle u_3' \rangle = \big|\langle \mathbf{u}' \rangle\big| \cos\theta'$ [with similar relationships for $\langle \bar{u}_i \rangle$, $i = 1, 2, 3$ in terms of its spherical coordinates $\big(|\langle \bar{\mathbf{u}} \rangle|, \bar{\theta}, \bar{\phi}\big)$]. Let the velocity vectors $\langle \bar{\mathbf{u}} \rangle$ and $\langle \mathbf{u}' \rangle$ have an angle γ between them. By trigonometry, the angle γ is related to the angles $\bar{\theta}$, $\bar{\phi}$ and θ', ϕ' as $\cos\gamma = \cos\bar{\theta}\cos\theta' + \sin\bar{\theta}\sin\theta'\cos(\bar{\phi} - \phi')$. For simplicity, let us write $Q \equiv \big|\langle \bar{\mathbf{u}} \rangle\big| = \big(\langle \bar{u}_j \rangle \langle \bar{u}_j \rangle\big)^{1/2}$ and $q \equiv \big|\langle \mathbf{u}' \rangle\big| = \big(\langle u_j' \rangle \langle u_j' \rangle\big)^{1/2}$. A result useful for the evaluation of the integrals displayed above is to apply the addition theorem for spherical harmonics (Matthews and Walker, 1970) to write the reciprocal of the magnitude of the spatially filtered total instantaneous velocity vector $\big|\langle \mathbf{u} \rangle\big| = \big|\langle \bar{\mathbf{u}} \rangle + \langle \mathbf{u}' \rangle\big|$ as [recalling that $(Q, \bar{\theta}, \bar{\phi})$ and (q, θ', ϕ') are the spherical coordinates of $\langle \bar{\mathbf{u}} \rangle$ and $\langle \mathbf{u}' \rangle$, respectively]

$$\frac{1}{\big|\langle \bar{\mathbf{u}} \rangle + \langle \mathbf{u}' \rangle\big|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{(2l+1)} \frac{q_<^l}{q_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\bar{\theta}, \bar{\phi}), \qquad (74)$$

where $Y_{lm}(\theta, \phi)$ denotes the spherical harmonics (orthonormal functions defined over the unit sphere),[19] $*$ denotes the complex conjugation operation, and $(q_<, q_>) = (q, Q)$ or (Q, q) depending on which of q

[19] A brief derivation of Equation (74) follows. For definiteness, we assume that $\big|\langle \mathbf{u}' \rangle\big| \equiv q < \big|\langle \bar{\mathbf{u}} \rangle\big| \equiv Q$. Then, introducing $s = q/Q$ we have

$$\frac{1}{\big|\langle \bar{\mathbf{u}} \rangle + \langle \mathbf{u}' \rangle\big|} = \frac{1}{\big(Q^2 + q^2 + 2Qq\cos\gamma\big)^{1/2}} = \frac{1}{Q}\big(1 + s^2 + 2s\cos\gamma\big)^{-1/2}.$$

Observe $\cos\gamma = -\cos(\pi - \gamma)$ where $\pi - \gamma$ is the angle between the vectors $\langle \bar{\mathbf{u}} \rangle$ and $-\langle \mathbf{u}' \rangle$. Noting that the vector $-\langle \mathbf{u}' \rangle$ has the spherical coordinates $(q, \pi - \theta', \pi + \phi')$, recognizing that the equation above is the generating function for the Legendre polynomials, and using the addition theorem for spherical harmonics (Matthews and Q is larger (viz., $q_<$ denotes the smaller of q and Q, whereas $q_>$ denotes the larger of q and Q). Equation (74) gives $\left|\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\right|^{-1}$ in a completely factorized form in the 'coordinates' $\langle\bar{\mathbf{u}}\rangle = (Q, \bar{\theta}, \bar{\phi})$ and $\langle\mathbf{u}'\rangle = (q, \theta', \phi')$. This is convenient for any integration over the probability density function of $\langle\mathbf{u}'\rangle$ where one variable is the variable of integration (i.e., $\langle\mathbf{u}'\rangle$) and the other is the 'coordinate' of a fixed point (i.e., $\langle\bar{\mathbf{u}}\rangle$).[20] The price paid is that there is a double summation involved, rather than a single term.

Now, to evaluate the integrals of Equations (71), (72), and (73), we rewrite[21]

$$\begin{aligned}\left|\langle\mathbf{u}\rangle\right| \equiv \big(\langle u_j\rangle\langle u_j\rangle\big)^{1/2} &= \Big(\left|\langle\bar{\mathbf{u}}\rangle\right|^2 + \left|\langle\mathbf{u}'\rangle\right|^2 + 2\left|\langle\bar{\mathbf{u}}\rangle\right|\left|\langle\mathbf{u}'\rangle\right|\cos\gamma\Big) \\ &\quad \times \frac{1}{\left|\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\right|} \\ &= \Big(Q^2 + q^2 + 2Qq\cos\gamma\Big) \times \frac{1}{\left|\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\right|}. \qquad (75)\end{aligned}$$

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and Walker, 1970), we get

$$\begin{aligned}\frac{1}{\left|\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\right|} &= \frac{1}{Q}\sum_{l=0}^{\infty} s^l P_l\big(\cos(\pi-\theta)\big) \\ &= \sum_{l=0}^{\infty}\frac{q^l}{Q^{l+1}}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\pi-\theta', \pi+\phi')Y_{lm}(\bar{\theta},\bar{\phi}) \\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{q^l}{Q^{l+1}}Y^*_{lm}(\pi-\theta', \pi+\phi')Y_{lm}(\bar{\theta},\bar{\phi}) \\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(-1)^m}{2l+1}\frac{q^l}{Q^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\bar{\theta},\bar{\phi}).\end{aligned}$$

Here, $P_l(z)$ is the Legendre polynomial of degree l. Clearly if $Q < q$, we should expand in terms of the ratio Q/q. If we use $q_<$ to denote the smaller and $q_>$ to denote the larger of q and Q, then the above equation can be written as Equation (74).

[20] For example, Equations (71) to (73) involve computing various statistical quantities by averaging their effect over all possible values of the random velocity $\langle\mathbf{u}'\rangle$ with the time-averaged, spatially-averaged velocity $\langle\bar{\mathbf{u}}\rangle$ prescribed to be some fixed quantity.

[21] Note that

$$\begin{aligned}\left|\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\right|^2 &= \big(\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\big)\cdot\big(\langle\bar{\mathbf{u}}\rangle + \langle\mathbf{u}'\rangle\big) \\ &= \Big(\left|\langle\bar{\mathbf{u}}\rangle\right|^2 + \left|\langle\mathbf{u}'\rangle\right|^2 + 2\langle\bar{\mathbf{u}}\rangle\cdot\langle\mathbf{u}'\rangle\Big),\end{aligned}$$

where $\cdot$ denotes the scalar product. Also, recall that by definition of the scalar product $\langle\bar{\mathbf{u}}\rangle\cdot\langle\mathbf{u}'\rangle = \left|\langle\bar{\mathbf{u}}\rangle\right|\left|\langle\mathbf{u}'\rangle\right|\cos\gamma$, where γ is the angle between the vectors $\langle\bar{\mathbf{u}}\rangle$ and $\langle\mathbf{u}'\rangle$.

If we substitute Equations (74) and (75) in Equations (71), (72) and (73), we obtain the following explicit results:[22]

$$\overline{|\langle\mathbf{u}\rangle|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{(2l+1)} \Bigg\{ \Big(Q^2 a_{lm}^{(1)[-l-1]_+} + a_{lm}^{(1)[-l+1]_+} + 2Q a_{lm}^{(2)[-l]_+}\Big) Y_{lm}(\bar\theta,\bar\phi) Q^l + \Big(Q^2 a_{lm}^{(1)[l]_-} + a_{lm}^{(1)[l+2]_-} + 2Q a_{lm}^{(2)[l+1]_-}\Big) \frac{Y_{lm}(\bar\theta,\bar\phi)}{Q^{l+1}} \Bigg\}; \quad (76)$$

$$\overline{|\langle\mathbf{u}\rangle|\langle u_i'\rangle} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{(2l+1)} \Bigg\{ \Big(Q^2 a_{lm;i}^{(3)[-l-1]_+} + a_{lm;i}^{(3)[-l+1]_+} + 2Q a_{lm;i}^{(4)[-l]_+}\Big) Y_{lm}(\bar\theta,\bar\phi) Q^l + \Big(Q^2 a_{lm;i}^{(3)[l]_-} + a_{lm;i}^{(3)[l+2]_-} + 2Q a_{lm;i}^{(4)[l+1]_-}\Big) \frac{Y_{lm}(\bar\theta,\bar\phi)}{Q^{l+1}} \Bigg\}; \quad (77)$$

and

$$\overline{|\langle\mathbf{u}\rangle||\langle\mathbf{u}'\rangle|^2} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(-1)^m}{(2l+1)} \Bigg\{ \Big(Q^2 a_{lm}^{(1)[-l+1]_+} + a_{lm}^{(1)[-l+3]_+} + 2Q a_{lm}^{(2)[-l+2]_+}\Big) Y_{lm}(\bar\theta,\bar\phi) Q^l + \Big(Q^2 a_{lm}^{(1)[l+2]_-} + a_{lm}^{(1)[l+4]_-} + 2Q a_{lm}^{(2)[l+3]_-}\Big) \frac{Y_{lm}(\bar\theta,\bar\phi)}{Q^{l+1}} \Bigg\}. \quad (78)$$

The coefficients in Equations (76), (77), and (78) are:

$$a_{lm}^{(1)[r]_{-,+}} = \int_{B_{-,+}(Q)} Y_{lm}^*(\theta',\phi') |\langle\mathbf{u}'\rangle|^r \Psi(\langle\mathbf{u}'\rangle)\, d\langle\mathbf{u}'\rangle; \quad (79)$$

$$a_{lm}^{(2)[r]_{-,+}} = \int_{B_{-,+}(Q)} Y_{lm}^*(\theta',\phi') |\langle\mathbf{u}'\rangle|^r \cos\gamma\ \Psi(\langle\mathbf{u}'\rangle)\, d\langle\mathbf{u}'\rangle; \quad (80)$$

$$a_{lm;i}^{(3)[r]_{-,+}} = \int_{B_{-,+}(Q)} Y_{lm}^*(\theta',\phi') |\langle\mathbf{u}'\rangle|^r \langle u_i'\rangle \Psi(\langle\mathbf{u}'\rangle)\, d\langle\mathbf{u}'\rangle; \quad (81)$$

$$a_{lm;i}^{(4)[r]_{-,+}} = \int_{B_{-,+}(Q)} Y_{lm}^*(\theta',\phi') |\langle\mathbf{u}'\rangle|^r \langle u_i'\rangle \cos\gamma\ \Psi(\langle\mathbf{u}'\rangle)\, d\langle\mathbf{u}'\rangle; \quad (82)$$

[22] The series in Equation (74) is uniformly and absolutely convergent, so the summation and integration operations may be exchanged by the Lebesgue theorem on dominated convergence (Spiegel, 1969).

Here, $B_-(Q) \equiv \{\langle \mathbf{u}' \rangle : |\langle \mathbf{u}' \rangle| < |\langle \bar{\mathbf{u}} \rangle| \equiv Q\}$ and $B_+(Q) \equiv \{\langle \mathbf{u}' \rangle : |\langle \mathbf{u}' \rangle| > |\langle \bar{\mathbf{u}} \rangle| \equiv Q\}$ are regions in the velocity fluctuation space inside and outside, respectively, a sphere of radius Q.

To complete the evaluation of $\bar{f}_i$ and F, we require a specification for the velocity PDF $\Psi(\langle \mathbf{u}' \rangle)$. To proceed, we must recognize that $\Psi(\langle \mathbf{u}' \rangle)$ is not something "physically real" and "absolute", but rather an encoding of a certain *state of knowledge* about the range of possible values of the spatially filtered velocity fluctuations. The impoverished framework of the k-ϵ model provides predictions only of the first and second moments of $\langle u_i' \rangle$ [the latter of which are obtained using the Boussinesq eddy viscosity approximation of Equation (23)]. Given this (necessarily) incomplete state of knowledge about $\langle u_i' \rangle$, it is logically desirable to assign $\Psi(\langle \mathbf{u}' \rangle)$ in accordance with the maximum entropy principle (MAXENT). MAXENT probability assignments (Jaynes, 1982) have a number of intuitively appealing interpretations. The maximum entropy distribution is the safest, most 'conservative' distribution to use for prediction because it spreads the probability out over the full range of *possible* values of the random variable (e.g., $\langle u_i' \rangle$) that are consistent with our given information or constraints (e.g., prescribed mean values $\overline{\langle u_i' \rangle} = 0$, and Reynolds stresses $\overline{\langle u_i' \rangle \langle u_j' \rangle}$) and, in doing so, prevents us from making arbitrary assumptions not justified by our information.

In the current problem, where our state of knowledge is limited to the first and second moments of $\langle u_i' \rangle$, the maximization of the entropy functional selects among all the possible probability distributions satisfying constraints imposed by the given first and second moments of the spatially filtered velocity fluctuations, the Gaussian PDF with the form

$$\Psi(\langle \mathbf{u}' \rangle) = \frac{1}{(2\pi)^{3/2}\sqrt{\det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2} \langle \mathbf{u}' \rangle^T \mathbf{\Sigma}^{-1} \langle \mathbf{u}' \rangle \right), \qquad (83)$$

where $\det(\cdot)$ denotes determinant, T is the matrix transpose operation, and $\mathbf{\Sigma} \equiv \overline{\langle u_i' \rangle \langle u_j' \rangle}$ is the covariance matrix of the spatially filtered velocity fluctuations (Reynolds stress tensor). With $\Psi(\langle \mathbf{u}' \rangle)$ given by Equation (83) [elliptically symmetric distribution], the integrals defining the coefficients in Equations (79) to (82) can be evaluated straightforwardly by transforming $\Psi(\langle \mathbf{u}' \rangle)$ to a spherically symmetric distribution through the diagonalization of $\mathbf{\Sigma}$ (assumed to be positive definite), and then evaluating the resulting integrals in spherical coordinates with infinitessimal volume element $d\langle \mathbf{u}' \rangle = |\langle \mathbf{u}' \rangle|^2 \sin\theta' \, d|\langle \mathbf{u}' \rangle| d\theta' \, d\phi' \equiv |\langle \mathbf{u}' \rangle|^2 \, d|\langle \mathbf{u}' \rangle| \, d\Omega'$ (where $d\Omega' \equiv \sin\theta' \, d\theta' \, d\phi'$ is the differential solid angle with units of steradians).

To illustrate the application of Equations (76) to (82) to the evaluation of $\bar{f}_i$ and F, consider the simple example of isotropic Gaussian turbulence with the covariance matrix of the spatially filtered velocity fluctuations in Equation (83) specified as $\mathbf{\Sigma} \equiv \overline{\langle u_i' \rangle \langle u_j' \rangle} = \frac{2}{3}\overline{\kappa}\mathbf{I}$ [where $\mathbf{I}$ is the (3×3) identity matrix]. Let us calculate the purely 'isotropic' contribution to $\bar{f}_i$ and F arising from the $l = 0$, $m = 0$ term in Equations (76) to (78). Noting that $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$, it is straightforward to evaluate the coefficients of Equations (79) to (82) for $l = m = 0$. Because $\Psi(\langle \mathbf{u}' \rangle)$ depends only on $|\langle \mathbf{u}' \rangle|$, the integral over the direction (or, angular part) of $\langle \mathbf{u}' \rangle$ involves only elementary trigonometric functions and can be carried out easily to give

$$a_{00}^{(1)[r]-,+} = 2\sqrt{\pi} A I_{-,+}(r+2; Q), \quad r = -1, 0, 1 \ldots; \tag{84}$$

$$a_{00}^{(2)[r]-,+} = 0, \quad r = -1, 0, 1 \ldots; \tag{85}$$

$$a_{00;i}^{(3)[r]-,+} = 0, \quad r = -1, 0, 1 \ldots; \tag{86}$$

$$a_{00;i}^{(4)[r]-+} = \sqrt{\pi} A \left(\frac{2}{3} \frac{\langle \bar{u}_i \rangle}{Q} \right) I_{-,+}(r+3; Q), \quad r = -1, 0, 1, \ldots. \tag{87}$$

Here, $A \equiv 3\sqrt{3}/8(\pi\overline{\kappa})^{3/2}$. Furthermore,

$$I_-(n; Q) \equiv \int_0^Q |\langle \mathbf{u}' \rangle|^n \exp\Big(-\alpha |\langle \mathbf{u}' \rangle|^2\Big)\, d|\langle \mathbf{u}' \rangle|, \quad n = 0, 1, 2, \ldots \tag{88}$$

and

$$I_+(n; Q) \equiv \int_Q^\infty |\langle \mathbf{u}' \rangle|^n \exp\Big(-\alpha |\langle \mathbf{u}' \rangle|^2\Big)\, d|\langle \mathbf{u}' \rangle|, \quad n = 0, 1, 2, \ldots, \tag{89}$$

with $\alpha \equiv 3/(4\overline{\kappa})$. The integrals over the 'radial' part $|\langle \mathbf{u}' \rangle|$ in Equations (88) and (89) can be evaluated[23] to give

$$\begin{aligned} I_-(n; Q) &= \frac{1}{2} \alpha^{-(n+1)/2} \gamma\big((n+1)/2; \alpha Q^2\big) \\ &= \frac{1}{2} \alpha^{-(n+1)/2} \gamma\big((n+1)/2; 3Q^2/4\overline{\kappa}\big) \end{aligned} \tag{90}$$

and

$$\begin{aligned} I_+(n; Q) &= \frac{1}{2} \alpha^{-(n+1)/2} \Gamma\big((n+1)/2; \alpha Q^2\big) \\ &= \frac{1}{2} \alpha^{-(n+1)/2} \Gamma\big((n+1)/2; 3Q^2/4\overline{\kappa}\big), \end{aligned} \tag{91}$$

[23] Put $|\langle \mathbf{u}' \rangle| = (t/\alpha)^{1/2}$ in Equations (88) and (89), so $d|\langle \mathbf{u}' \rangle| = \frac{1}{2}\alpha^{-1/2} t^{-1/2}\, dt$.

where $\gamma(\nu; x)$ and $\Gamma(\nu; x)$ are, respectively, the incomplete gamma function and complementary incomplete gamma function.[24]

We derive an explicit form for $\bar{f}_i$ and F arising from the purely 'isotropic' contribution provided by the $l = 0$, $m = 0$ spherical harmonic mode. To this end, substitute Equations (84) to (91) into Equations (76) to (78) for the $(l, m) = (0, 0)$ term, and insert these results into Equations (69) and (70). This gives finally (on using the definitions of α and A)

$$\bar{f}_i \approx -C_D \hat{A} Q \langle \bar{u}_i \rangle \left[\frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}; \frac{3Q^2}{4\overline{\kappa}}\right) + \frac{3}{\sqrt{\pi}} \frac{Q}{\overline{\kappa}^{1/2}} \Gamma\left(1; \frac{3Q^2}{4\overline{\kappa}}\right) + \frac{16}{9\sqrt{\pi}} \frac{\overline{\kappa}}{Q^2} \left(\frac{3\sqrt{\pi}}{4} + \frac{1}{2} \gamma\left(\frac{5}{2}; \frac{3Q^2}{4\overline{\kappa}}\right)\right)\right] \tag{92}$$

and

$$F \approx -C_D \hat{A} Q \overline{\kappa} \left[\frac{4}{3} + \frac{20\sqrt{3}}{9\sqrt{\pi}} \frac{Q}{\overline{\kappa}^{1/2}} \Gamma\left(2; \frac{3Q^2}{4\overline{\kappa}}\right) + \frac{16\sqrt{3}}{9\sqrt{\pi}} \frac{\overline{\kappa}^{1/2}}{Q} \Gamma\left(3; \frac{3Q^2}{4\overline{\kappa}}\right) + \frac{8}{9\sqrt{\pi}} \gamma\left(\frac{5}{2}; \frac{3Q^2}{4\overline{\kappa}}\right) + \frac{32}{9\sqrt{\pi}} \frac{\overline{\kappa}}{Q^2} \gamma\left(\frac{7}{2}; \frac{3Q^2}{4\overline{\kappa}}\right)\right]. \tag{93}$$

Interestingly, the purely 'isotropic' contribution from the $(l, m) = (0, 0)$ spherical harmonic mode to F appears as a sink in the transport equation for $\overline{\kappa}$ and represents, therefore, a transfer of turbulence energy from the resolved scales (i.e., scales larger than the filter width Δ) to the sub-filter motions. However, for other spherical harmonic modes (viz., for $l > 0$ and $|m| \geq 0$) there can be *backscatter*, i.e. transfer of turbulence energy from the sub-filter motions to the resolved scales of the spatially-filtered velocity fluctuations implying terms in F that appear as a source in the $\overline{\kappa}$-equation.

[24] The incomplete gamma function $\gamma(\nu; x)$ and its complementary cohort $\Gamma(\nu; x)$ are defined by the integral representations (Spanier and Oldham, 1987)

$$\gamma(\nu; x) = \int_0^x t^{\nu-1} \exp(-t)\, dt, \qquad x \geq 0,\ \nu > 0$$

and

$$\Gamma(\nu; x) = \int_x^\infty t^{\nu-1} \exp(-t)\, dt, \qquad x \geq 0,\ \nu > 0.$$

These two functions sum to the complete gamma function $\Gamma(\nu)$, viz.

$$\gamma(\nu; x) + \Gamma(\nu; x) = \Gamma(\nu).$$

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