The Use of the Primitive Equations of Motion in Numerical Prediction

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Abstract

An obstacle to the use of the primitive hydrodynamical equations for numerical pre-
diction is that the initial wind and pressure fields determined by conventional means give
rise to spurious large-amplitude inertio-gravitational oscillations which obscure the meteoro-
logically significant large-scale motions. It is shown how this difficulty may be overcome
by the use of a relationship between wind and pressure which enables one to determine
these fields in such a manner that the *noise* motions do not arise. The method is illustrated
by a numerically computed example.
The wind-pressure relationship is in a sense a generalization of the geostrophic approxi-
mation and may be used where the latter approximation is inapplicable, either to determine
initial conditions or to derive a set of *filtering* equations for numerical prediction analogous
to the quasi-geostrophic equations.

I. Introduction

The recent widespread revival of interest
in numerical weather prediction has been
brought about partly by the development of
high-speed electronic computing apparatus
but also, to a large extent, by the very en-
couraging early results obtained through
systematic use of the geostrophic approxima-
tion in what has come to be known as the
quasi-geostrophic equations of motion. In
these equations it is assumed that the wind is
and remains nearly geostrophic, at least
enough so that the horizontal acceleration
terms in the equations of motion may be
approximated by the accelerations of the
geostrophic wind. Numerical integrations of

these equations have already yielded forecasts
comparable in accuracy to those produced by
conventional methods, and probably superior
in accuracy in the case of new developments.
In purely theoretical investigations the use of
the geostrophic approximation, by automa-
tically filtering out high-frequency "noise"
disturbances, has aided greatly in the under-
standing of the physical properties of large-
scale flow systems where the high-frequency
inertial effects are unimportant.

Yet those who have participated actively
in this work are aware that this is barely the
beginning of a new era, an era in which
numerical methods will find increasing applica-
tion in practical and theoretical meteorology,
and that the quasi-geostrophic equations are
but a first approximation to the equations
which will eventually be used to describe the
atmospheric motion. Experience has already

revealed systematic errors attributable to the geostrophic approximation. The approximation tends to displace depressions too tar northward over the United States, and does not, apparently, permit the development of frontal discontinuities. It is of course possible to arrive at higher approximations by successive iteration of the geostrophic approximation, but this process, besides leading to severe mathematical complications, is valid only when the geostrophic deviations are small in the first place. Moreover, the time order of the equations is increased by one with each iteration, and since the time order cannot exceed *three*, it is evident that the sequence of approximations would not be convergent.

In any case it is likely that in the process the simplicity and elegance of the geostrophic approximation would be destroyed, and thereby also the reason for its introduction. One is therefore led to reexamine the possibility of using the primitive Eulerian equations, as Richardson[1] first proposed.

The difficulty here, as I have pointed out in previous papers, is that the initial values of wind and pressure cannot as a rule be prescribed independently with sufficient accuracy. One is apparently forced again to introduce for the wind values either the geostrophic approximation or something very little better. As will be shown later this not only introduces an initial inaccuracy but gives rise to spurious inertio-gravitational oscillations which obscure the meteorologically significant motions. The purpose of the present note is to show that accurate initial winds can be determined from the pressure field alone, and that if this is done inertio-gravitational oscillations will not arise.

II. Characterization of large-scale atmospheric motions

We shall deal with statically stable motions in which the horizontal scale far exceeds the vertical scale. It is therefore permissible to assume that the motions are in quasi-hydrostatic equilibrium. If f is the coriolis parameter $2\Omega \sin$ (latitude), V a characteristic horizontal velocity, and S a characteristic horizontal scale distance, the ratio of the inertial force to the coriolis force is in order of magnitude the non-dimensional Rossby number

[1] Weather Prediction by Numerical Process, Cambridge, 1922.

V/fS. The motion is said to be quasi-geostrophic if this number is small compared to unity. There are, however, cases where either because V is large or S is small the Rossby number is not small. Nevertheless the large-scale meteorologically significant motions are distinguishable from the inertio-gravitational oscillations, the other class of possible quasi-hydrostatic motions. We shall arrive at the required method for determining the horizontal wind field by asking for their distinguishing characteristics.

There are essentially two types of small amplitude wave perturbation of an atmosphere moving with constant angular velocity. They are distinguished by their frequencies and speeds of propagation. The meteorologically important type is characterized by small frequencies and by velocities of propagation of the same order of magnitude as the speed of the imbedding current. The second type has frequencies in excess of f, periods less than a half pendulum day, and velocities of propagation of the order of $(gH^2\partial \ln \theta/\partial z)^{1/2}$, where H is the vertical scale height of the disturbance and θ is the potential temperature. For the large scale disturbances of the atmosphere $H \sim 10^4$ m and $\partial \ln \theta/\partial z \sim 10^{-5}$ m^{-1}. Hence $(gH^2\partial \ln \theta/\partial z)^{1/2} \sim$

$$\sim \left(\frac{1}{10}gH\right)^{1/2} \sim 100 \text{ m/sec}^{-1},$$ which is much

greater than the speed of the wind in the troposphere and very much greater than the speed of propagation of the observed large-scale systems. Even when the motions are of large amplitude and therefore not linearly superposable solutions of perturbation equations, a distinction can be drawn between the two types, at least in orders of magnitude of frequency and velocity of propagation.

From what can be inferred from available observations the bulk of the energy in the troposphere is confined to the first class of motion. This partition of energy may be attributed to the circumstances that the principle energy sources in the atmosphere have characteristic periods large compared to the pendulum day and therefore excite little of the second type of motion and that the first type is stable with respect to whatever perturbations of the second type are excited, either directly by the external sources of energy or indirectly by nonlinear interaction with the first type.

III. The balance equations

Let us now examine the order of magnitude of the horizontal divergence in the meteorologically important systems—systems that are characterized by particle velocities and speeds of propagation small compared to the gravity-inertia speed $(gH^2\partial \ln \theta/\partial z)^{1/2}$. If a flow is quasi-geostrophic it can be shown by the use of dimensional arguments[1] that the ratio of the horizontal divergence $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$ to one of its component terms, say $\partial u/\partial x$, is just the Rossby number and therefore this is small. In the case where the Rossby number is not small the ratio is found to be $V^2/gH^2\partial \ln \theta/\partial z$ and is therefore again small. Thus the significant motions of the free atmosphere are characterized by small horizontal divergences, and we may therefore express u and v in terms of a stream function:

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}.$$

It may further be shown that the convective acceleration terms involving w in the horizontal equations of motion are smaller in order of magnitude than the remaining convective acceleration terms. If we omit these small terms, substitute the stream function expressions for u and v, and take the horizontal divergence of the equations of motion, we obtain, when pressure is used as the vertical coordinate,

$$\nabla^2\psi + \nabla\psi \cdot \nabla f - \frac{2}{f}\left[\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 - \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}\right] = \frac{\nabla^2\varphi}{f}$$

where φ is the geopotential. The above equation, which shall be called the *balance equation*, suffices to determine the horizontal velocity field if the field of geopotential is known. This equation is of the Monge-Ampere type and can be solved for ψ as a boundary value problem providing

$$\frac{\nabla^2\varphi}{f} + \frac{f}{2} > 0$$

i.e., if the geostrophic relative vorticity $\dfrac{\nabla^2\varphi}{f}$ is greater than $-f/2$. If the non-linear terms are small we obtain the geostrophic approximation

by integration. In this sense the balance equation is a generalization of the geostrophic approximation. Both determine the velocity from the geopotential.

Now it is characteristic of the inertio-gravitational motion that the horizontal divergence is not relatively small. Hence if the u and v, which must be prescribed independently in the primitive quasi-static equations of motion, are determined from the balance equations, the inertio-gravitational motions are automatically excluded initially, and since they are not generated to any appreciable extent during the motion, they will never appear.

The balance equation has been derived by Fjørtoft (unpublished work) in an independent investigation as a necessary condition for the stability of the non-divergent flow with respect to perturbations with horizontal divergence. The neglect of the divergence terms in the divergence equation has been justified empirically by S. PETTERSSON[1] who finds them to be one to two orders of magnitude smaller than the remaining terms.

IV. Illustrations of the use of the balance equation

I had thought at one time that the use of geostrophic u's and v's as part of the initial specification of the flow would lead at most to small amplitude high-frequency inertio-gravitational fluctuations superposed on an essentially correct low-frequency trend motion. However a recent numerical integration of the non-linear Eulerian equations for a barotropic atmosphere carried out in collaboration with B. Bolin led to the suspicion that large amplitude inertio-gravitational perturbations can and do arise. In retrospect this is not surprising, for although the initial divergence of the geostrophic wind is zero or nearly zero, the first time derivative will not be zero since the balance equation will not be satisfied. The experiment was not conclusive because the fluctuations could have been due in part to computational instabilities arising from incorrect boundary conditions. (The boundaries were geometrical surfaces immersed in the fluid.) To investigate the matter further and to verify the filtering properties of the balance

[1] Charney, J. G., Geof. Publ. 17, No. 2, 1948.

[1] Tellus, 5, No. 3, 1953.

equation the following numerical experiment was recently performed in Princeton:

Initial conditions were prescribed for the flow of an atmosphere consisting of two homogeneous incompressible layers of different density bounded above and below by rigid surfaces. The lower layer was assumed to be so deep that its motion could be ignored. The upper layer would then move as though it were a single layer with a free surface but with gravity reduced in the ratio $\left(1 - \dfrac{\varrho'}{\varrho}\right)$, ϱ' and ϱ being the densities of the upper and lower layers respectively. This was done to simulate the actual static stability of the atmosphere. The mean thickness of the upper layer was taken to be H and $\left(1 - \dfrac{\varrho'}{\varrho}\right)$ was chosen to equal $H\partial \ln \theta/\partial z$ for the actual atmosphere, *i.e.*, about 10^{-1}. In this way the buoyancy forces, as well as the speed of the internal gravity waves, became comparable to those in the actual atmosphere. The coriolis parameter f was taken to be constant and was set equal to 10^{-4} sec^{-1} corresponding to a latitude of about $43°$. The motion was assumed to take place in the x, y plane between the rigid lateral walls $y = 0$, $y = D$.

The initial velocity field was prescribed by the periodic stream function

$$\psi = A \sin \frac{2\pi x}{L} \sin \frac{\pi y}{D}$$

with $L = 4{,}000$ km and $D = 2{,}000$ km and $u_{max} = v_{max} = 2\pi A/L = 50$ m sec^{-1}. To check first that the balance equation did indeed exclude inertio-gravitational motions the initial values of the thickness field h were determined from the balance equation, which in this case took the form

$$\nabla^2 \psi - \frac{2}{f}\left[\left(\frac{\partial^2 \psi}{\partial x \partial y}\right)^2 - \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2}\right] = \frac{\nabla^2 \varphi}{f}$$

where $\qquad \varphi = g\left(1 - \dfrac{\varrho'}{\varrho}\right) h.$

Since v, the y-component of velocity, is zero at $y = 0$ and $y = D$, the second equation of motion

$$\frac{dv}{dt} + fu = -\frac{\partial \varphi}{\partial y}$$

yielded the side condition

$$fu = -\frac{\partial \varphi}{\partial y}$$

at $y = 0$ and $y = D$. The solution of the balance equation subject to this condition was found to be

$$\varphi = g\left(1 - \frac{\varrho'}{\varrho}\right) H + Af \sin \frac{2\pi x}{L} \sin \frac{\pi y}{D} +$$

$$+ \frac{A^2}{4}\left(\frac{\pi^2}{D^2}\cos\frac{4\pi x}{L} + \frac{4\pi^2}{L^2}\cos\frac{2\pi y}{D}\right).$$

It is easy to show that if the divergence had vanished exactly the motion would have been exactly stationary. Since the use of the balance equation was supposed to prevent the occurrence of large divergences it was to be expected that also in the present case the motion would remain nearly stationary. This was found to be so for a forty-eight hour period to within the limits of a quite small computational error.

A numerical integration was next performed using the geostrophic initial conditions

$$\varphi = f\psi + g\left(1 - \frac{\varrho'}{\varrho}\right) H, \quad u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x},$$

KINETIC ENERGY
(Initial velocities satisfy balance condition)

KINETIC ENERGY
(Initial velocities geostrophic)

POTENTIAL ENERGY
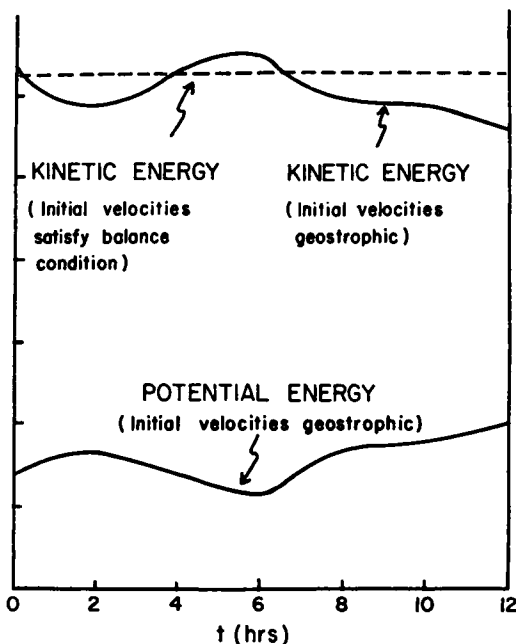(Initial velocities geostrophic)

Fig. 1. Variations in time of kinetic and potential energy.

which of course also satisfy the boundary conditions $fu = -\partial\varphi/\partial y$ at $y = 0$ and $y = D$. In this case large divergences ocurred which in turn produced large potential energy changes according to the formula

$$\frac{dP}{dt} = \frac{\varrho}{2} \int\limits_{0}^{L} \int\limits_{0}^{D} g\left(1 - \frac{\varrho'}{\varrho}\right) \frac{h^2}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx dy$$

in which P is the potential energy per wave length. The kinetic energy was accordingly highly variable. Figure 1 shows the kinetic and potential energies plotted as functions of time. The upper curve represents the kinetic energy and the lower the potential energy. The horizontal line represents the constant kinetic energy in the non-divergent and balanced cases.

We note that there is a fluctuation of some 14 % in kinetic energy. The local fluctuations were considerably larger than this and must be regarded as errors.

The foregoing example indicates that the balance condition is far more suitable for determining the initial wind pressure fields than the geostrophic condition.

It should be remarked finally that the preservation of the velocity-pressure adjustment demanded by the balance equation necessarily entails horizontal divergences. I have merely attempted to show that these divergences must remain so small that the motion is approximately non-divergent at all times. Because the balance condition always holds, it is possible in principle to employ it in the same manner as the geostrophic approximation. It may be taken together with the hydrostatic equation, the vorticity equation, the physical equation and the continuity equation as determining the motion. The difficulty then is that it is not possible to combine these equations into a single equation in φ as was the case with the geostrophic approximation, and I have succeeded in solving them numerically only in the barotropic case (unpublished result).