A SIMPLE THREE-DIMENSIONAL MODEL FOR THE STUDY OF LARGE-SCALE EXTRATROPICAL FLOW PATTERNS

By Norman A. Phillips

University of Chicago[1,2]

(Manuscript received 31 May 1951)

ABSTRACT

A study is made of the hydrostatic and quasi-geostrophic motion of two superimposed layers of homogeneous and incompressible fluids of different densities, these fluids being contained between two rigid, horizontal plates. It is found that the local time derivatives of the pressure heights in the two layers and the height of their interface can be determined from partial differential equations similar to those developed by Charney for the equivalent-barotropic model.

The possibility of using this two-layer model to represent motions of a continuously stratified, baroclinic troposphere is explored by comparing the behavior of small perturbations superimposed on a zonal current in the two-layer model with the results of the continuous baroclinic perturbation theories of Eady and Fjørtoft. The remarkable similarity of behavior of the two-layer and the continuous perturbation models, which appears from this comparison, suggests that if the initial flow patterns of the two-layer model are determined from the initial flow patterns of the troposphere in a specified manner the later flow patterns in the troposphere can be inferred from the forecast flow patterns of the two-layer model.

This hypothesis is subjected to a preliminary test by computing the instantaneous sea-level pressure tendencies and vertical motions (in the middle troposphere) at the beginning of the severe storm of 24–25 November 1950 over eastern North America. The order of magnitude of the predicted quantities and their general distribution agree in many respects with the observed pressure changes and hydrometeors, but some disagreement exists. It is suggested that a part of this disagreement may be due to the effect of large normal accelerations on the validity of the quasi-geostrophic assumption.

1. Introduction

Many informative theoretical studies of the behavior of the large-scale flow patterns in extratropical latitudes have been made in the past decade.[3] A large number of these, beginning with the classical investigation by Rossby and collaborators [16], have taken cognizance of the fact that the "depth" of the atmosphere is much smaller than the horizontal scale of the large disturbances, and have treated the flow as two-dimensional and non-divergent. More recently, Charney [4] has demonstrated that the *mean* flow (mean with respect to pressure) can be described adequately in this manner when certain conditions are satisfied (the "equivalent barotropic" atmosphere). This has formed the basis for the partially successful 24-hr forecasts of the 500-mb flow pattern, which he and his co-workers have recently made by performing a numerical integration of the non-linear vorticity equation [6].

Although the non-divergent vorticity equation and the quasi-geostrophic assumption [3; 8] thus seem to provide a reasonably satisfactory method of forecasting the mean flow pattern, an adequate forecast of "weather," including precipitation, cloudiness and surface wind velocities, can be achieved only when the distribution of vertical motion and the "sea-level" pressure distribution are known. The equivalent barotropic atmosphere is patently incapable of providing this information, at least in the detail which is required. It is therefore necessary, as is recognized by Charney and his collaborators, to extend numerical forecasting techniques to atmospheric models which will permit the forecasting of vertical motion and the surface pressure field. The importance of baroclinicity in this respect has been emphasized by Sutcliffe [20].

Charney [4] has derived simplified equations which apply to the more general baroclinic case. They consist of the equation expressing the conservation of the vertical component of the absolute potential vorticity,

$$\frac{d}{dt}\left[\frac{1}{\rho}\frac{\partial\vartheta}{\partial z}(f+\zeta)\right] = 0, \tag{1}$$

and the hydrostatic equation,

$$\partial p/\partial z = -\rho g, \tag{2}$$

together with the assumption that the substantial derivative d/dt and the relative vorticity ζ are evaluated geostrophically. (ρ = density, ϑ = potential temperature, f = Coriolis parameter, p = pressure and g = acceleration of gravity.) These equations can be reduced to a single partial differential equation, in the three independent variables x, y and z, for the pressure tendency $\partial p/\partial t$. This equation can then be solved

[1] This paper is published as a contribution to a research project on the general circulation of the atmosphere, sponsored by the Office of Naval Research.

[2] Present affiliation: Institute for Advanced Study.

[3] A unified treatment of some of these studies has recently been made by Fjørtoft [10].

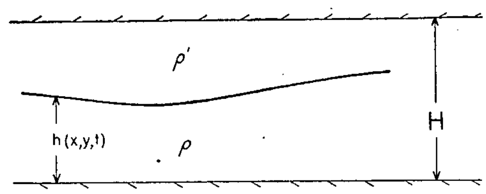

FIG. 1. Vertical cross section through two-layer model with rigid top.

numerically for $\partial p/\partial t$ by an extension of the method used for the two-dimensional barotropic model [6].

Although (1) and (2), together with the quasi-geostrophic assumption, represent a drastic simplification of the complete hydrodynamic equations as we know them, they may still include rather complicated motions when applied directly to the atmosphere. It may therefore be of some advantage to consider first a simple baroclinic model which obeys the same laws as those expressed in (1) and (2) but is simpler in its physical construction than is the actual atmosphere and therefore more easily understood. It is the purpose of this paper to construct such a model and to test its validity by applying it to an actual situation.

2. Description of the model

Probably the simplest model which contains some of the essential features of the baroclinic atmosphere is the following "two-layer" model. We consider two fluids, each homogeneous and incompressible, with densities ρ' and ρ ($\rho' < \rho$). They are contained between two rigid, horizontal plates, which are separated a distance H. The heavier fluid has a depth h and lies below the lighter fluid (fig. 1). h is a function of time t and the horizontal coordinates x and y. The potential vorticity equations for this model then take the form

$$\frac{d}{dt}\left(\frac{f+\zeta}{h}\right) = 0, \quad \frac{d'}{dt}\left(\frac{f+\zeta'}{H-h}\right) = 0, \qquad (3)$$

with the primed quantities referring to the lighter fluid. These equations contain implicitly the mass continuity equation and the kinematic boundary conditions at $z = 0$, $z = h$ and $z = H$. They also contain the assumption that the horizontal velocity in each fluid is independent of height, which in turn is based on hydrostatic balance and barotropy within each fluid.

Although the pressure p' at the upper rigid boundary cannot be "hydrostatic" in the sense that it is balanced by the weight of a column of air above it, and must be thought of as being balanced by a reaction of the rigid top, the presence of the rigid top does not affect the hydrostatic equation within the fluid, at least for the scale of motions in which we are interested. Before we write the hydrostatic equation, however, it will be convenient to define two variables z' and z,

$$z' = p'/\rho' g, \quad z = p_0/\rho g,$$

where p_0 is the pressure at the bottom. Apart from irrelevant constants, z' and z may be thought of as being the height of certain pressure surfaces in the upper and lower fluids. If we also define $\epsilon = \rho'/\rho < 1$, the hydrostatic equation takes the form

$$z = \epsilon z' + (1-\epsilon)h + \text{constant}. \qquad (4)$$

Following the procedure outlined in principle by Charney, we now evaluate the vorticity equations (3) geostrophically. With $\zeta' = f^{-1}g\,\nabla^2 z'$ and $\zeta = f^{-1}g\,\nabla^2 z$, they reduce to

$$\nabla^2\frac{\partial z}{\partial t} + \frac{f}{g}\,\boldsymbol{v}\cdot\nabla\eta - \frac{f\eta}{gh}\left(\frac{\partial h}{\partial t} + \boldsymbol{v}\cdot\nabla h\right) = 0, \qquad (5)$$

$$\nabla^2\frac{\partial z'}{\partial t} + \frac{f}{g}\,\boldsymbol{v}'\cdot\nabla\eta' + \frac{f\eta'}{g(H-h)}\left(\frac{\partial h}{\partial t} + \boldsymbol{v}'\cdot\nabla h\right) = 0, \qquad (6)$$

where the horizontal velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$, and the absolute vorticities η and η', are understood to be evaluated geostrophically.

Equation (4) may then be used to eliminate $\partial z/\partial t$ and $\partial z'/\partial t$, so that the following equation in $\partial h/\partial t$ results (note that $\epsilon\boldsymbol{v}'\cdot\nabla h = \boldsymbol{v}\cdot\nabla h$, geostrophically):

$$\nabla^2\frac{\partial h}{\partial t} - a(x,y)\frac{\partial h}{\partial t} + q(x,y) = 0, \qquad (7)$$

where

$$a(x,y) = \frac{f}{(1-\epsilon)g}\left[\frac{\eta}{h} + \epsilon\frac{\eta'}{H-h}\right], \qquad (8)$$

and

$$q(x,y) = \frac{f}{(1-\epsilon)g}\left[\boldsymbol{v}\cdot\nabla\eta - \epsilon\boldsymbol{v}'\cdot\nabla\eta' - \left(\frac{\eta}{h} + \frac{\eta'}{H-h}\right)\boldsymbol{v}\cdot\nabla h\right]. \qquad (9)$$

The behavior of the model is then computed in the following manner:

1. The functions $a(x, y)$ and $q(x, y)$ are evaluated geostrophically from the initial distribution of z', z and h (for example, at the points of a rectangular grid).

2. $\partial h/\partial t$ is determined by solving (7) numerically, using finite differences and assuming $\partial h/\partial t$ to be zero at the

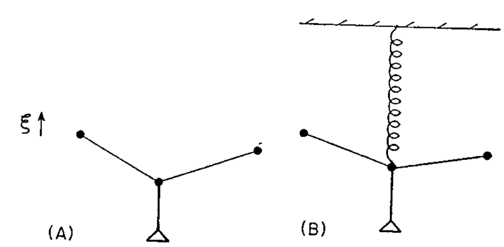

FIG. 2. Schematic diagram illustrating restraining influence of springs which must be added to elastic-net analogue of geostrophic vorticity equation when divergence and convergence are important.

lateral boundaries (which must then be sufficiently far removed from the region of interest).

3. $\partial h/\partial t$ is substituted in (5), which is solved numerically as a finite-difference "Poisson" differential equation for $\partial z/\partial t$.

4. The other unknown tendency, $\partial z'/\partial t$, is determined from (4) at each point, with use of the values of $\partial h/\partial t$ and $\partial z/\partial t$ already obtained.

5. The initial values of z', z and h are then extrapolated in time with use of the tendencies which have been calculated above.[4] The entire procedure is then repeated.

Charney [4] has used the mechanical analogue of a loaded elastic net to illustrate qualitatively the properties of the simple Poisson type of differential equation for the mean height tendency $\partial \bar{z}/\partial t$, which is used in the equivalent-barotropic model,

$$\nabla^2 \frac{\partial \bar{z}}{\partial t} + \frac{f}{g}\, \overline{\boldsymbol{v} \cdot \nabla \eta} = 0.$$

The known function $g^{-1} f \overline{\boldsymbol{v} \cdot \nabla \eta}$ can be considered as equivalent to a distribution of external forces $\sigma(x, y)$ (*e.g.*, weights or gas-filled balloons) acting vertically at the points of the elastic net (fig. 2, A), the neutral position of the net being horizontal ($\xi = 0$). The tendency $\partial \bar{z}/\partial t$ is then analogous to that vertical displacement ξ of the net which results when the external forces $\sigma(x, y)$ are balanced by the elastic restoring forces of the net (the latter being proportional to $\nabla^2 \xi$). The gross features of the field of $\partial \bar{z}/\partial t$ can thus be inferred from the gross distribution of the vorticity advection in the non-divergent model by considering the equivalent loading of the elastic net. A similar interpretation can also be given to the equation for $\partial h/\partial t$, which differs from a Poisson equation by the added term $-a(x, y)\, \partial h/\partial t$. If $a(x, y)$ is positive—the ordinary case—this modifies the analogue in the following manner. To each point of the net we attach one end of a linear spring, attaching the other end of the spring to a rigid support, the neutral position of the spring being $\xi = 0$ (fig. 2, B). The "compliance" of each spring should be proportional to the corresponding value of $a(x, y)$ at that point. It is clear from this analogue that the term $-a(x, y)\, \partial h/\partial t$ has a "damping" effect when $a(x, y) > 0$.

The role of the term $-a(x, y)\, \partial h/\partial t$ can also be seen by the following reasoning. For convenience, assume that $\boldsymbol{v} \cdot \nabla h = 0$ (this is not essential to the argument) and that $q(x, y)$ is positive. For example, consider $\boldsymbol{v} \cdot \nabla \eta$ and $\epsilon \boldsymbol{v}' \cdot \nabla \eta'$ both negative (wind blowing from high to low values of absolute vorticity), but $|\epsilon \boldsymbol{v}' \cdot \nabla \eta'| > |\boldsymbol{v} \cdot \nabla \eta|$. (Such a situation might exist to the east of a "cold" baroclinic trough.) In terms of

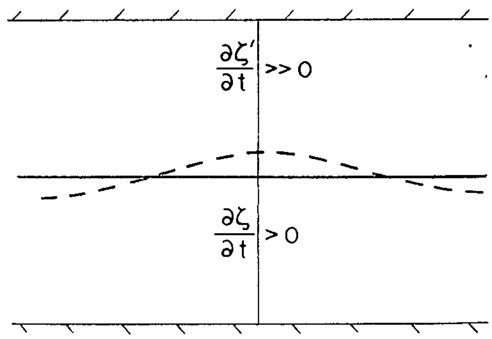

FIG. 3. Schematic cross section showing deformation (dashed line) of interface caused by greater advection of absolute vorticity in upper than in lower fluid.

the mechanical analogue, this corresponds to a positive force $\sigma(x, y)$, *i.e.*, a gas-filled balloon is attached to the point of the net under consideration.

As a first approximation, assume that the flow in each layer is non-divergent. The fact that $\epsilon \boldsymbol{v}' \cdot \nabla \eta'$ is more negative than $\boldsymbol{v} \cdot \nabla \eta$ would then imply that the local relative vorticity increase is greater in the top layer than in the bottom layer. For hydrostatic and geostrophic balance, this would require the internal boundary to be deformed as indicated by the dashed line in fig. 3, so that $\nabla^2(\partial z/\partial t) < \epsilon \nabla^2(\partial z'/\partial t)$. In turn, this must in general involve a tendency to raise h at the point in question, since we are not perfectly free to depress h arbitrarily in the surroundings. This lifting of the internal surface corresponds to convergence in the lower fluid and divergence in the upper fluid, *which tends to counteract the initial disparity in* $\boldsymbol{v} \cdot \nabla \eta$ *and* $\epsilon \boldsymbol{v}' \cdot \nabla \eta'$, and thus to reduce the magnitude of $\nabla^2(\partial h/\partial t)$ and $\partial h/\partial t$.

The equation for $\partial h/\partial t$ expresses the fact that a balance is reached between (a) the vertical motions (in addition to horizontal advection) necessary to make the pressure changes predicted from the quasi-geostrophic vorticity equations hydrostatically consistent, and (b) the simultaneous modification by these vertical motions of the pressure changes predicted from the quasi-geostrophic vorticity equations. This modification occurs via the divergence term in the vorticity equations.

The general nature of (7), with the vertical variation in horizontal advection of vorticity playing a prominent role, agrees well with the description of large-scale motions first proposed by Sutcliffe [20] and later modified by Sumner [19].

3. Comparison of model and continuous atmosphere

If this simple two-layer model is to be of any value in studying the behavior of flow patterns in the atmosphere, some method must be found to transform a given initial state of the atmosphere into an equivalent state of the model. Furthermore, after the model has developed for some time according to the method

[4] The conditions under which this extrapolation is "computationally stable" for the non-divergent model are discussed by Charney and others [6]. Similar restrictions undoubtedly apply to the model described above and will place an upper limit to the length of the time step which can be used for each successive time extrapolation.

outlined above, it must be possible to invert this transformation and obtain a forecast of the state of the atmosphere at this later time. In hydrodynamic laboratory experiments with physical models which are geometrically similar to the "prototype," this is accomplished by making certain non-dimensional numbers equal in the model and prototype. (For instance, in a problem of the type we are investigating, two such numbers would undoubtedly be a Richardson number, $gs(dU/dz)^{-2}$, and a Froude number, $U^2(gsH^2)^{-1}$, where s = vertical stability.) This is based on the fact that when the same types of forces act on the prototype and the *geometrically similar* model, and the differential equations and boundary conditions applicable to each of them are written in a non-dimensional form, these differential equations and boundary conditions will be identical when the boundaries are geometrically similar and the non-dimensional constant parameters, which appear as coefficients in the equations, are equal for model and prototype.

In our problem this most direct procedure is not possible, because the two-layer model is not geometrically similar to the atmospheric prototype. This lack of geometric similarity is a result of the fact that the continuous atmosphere has an infinite number of degrees of freedom, so to speak, in the vertical coordinate, while the two-layer model has only two such degrees of freedom, represented by z and z'. It is therefore clear that we cannot expect the two-layer model to simulate those motions of a continuous atmosphere which require many parameters to specify their distribution in the vertical. Fortunately, aerological evidence [11; 14] shows that the variation with height of the large-scale flow patterns of interest to us is rather simple. Thus, approximately, the vertical velocity-component (w) associated with these motions is zero at the ground, reaches a maximum value approximately half way between the ground and tropopause, and becomes small again in the vicinity of the tropopause. If we restrict our consideration to motions of the atmosphere which not only obey (1) and (2) but are also characterized by a simple variation with height, consistent with the distribution of vertical velocity just described, it might be possible to rewrite (1) and (2) so that they become formally similar to (3) and (4). (This procedure would be an extension of what Charney has done in defining the equivalent-barotropic model.) If these modified equations were then compared with the corresponding equations for the two-layer model, one would supposedly obtain a set of conditions under which the subsequent motions of the two-layer model and the "restricted" atmosphere would be geometrically similar. These conditions could be considered as defining equivalent Froude and Richardson numbers, *etc.*, or else as simply defining values of v, v', ϵ, h and H of the two-layer model in terms of the variables and parameters of the atmosphere.

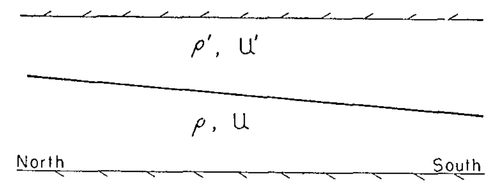

FIG. 4. Meridional cross section through basic current considered in study of linearized behavior of two-layer model.

A less direct method is carried out below. The behavior of a *continuous baroclinic model* with a rigid top is compared with that of the *two-layer model* with a rigid top, for small disturbances satisfying (1) and (2) or (3) and (4), these disturbances being superimposed on a basic, baroclinic zonal flow. Only disturbances of the continuous baroclinic model which have a distribution of w similar to that described above are considered. Application of conventional perturbation analysis to these two systems yields *frequency equations* which predict the behavior of the superimposed perturbations. If geometrically similar perturbations (*i.e.*, perturbations having the same wavelength) are to behave similarly in each system, it turns out that the basic flow pattern in the two-layer model must be determined from that of the continuous model in a certain manner. (By similar behavior of perturbations of equal wavelength in the two-layer and continuous models, we mean that the speed of propagation and instability properties of the waves are the same.) The hypothesis is then made that if an initial non-linear flow pattern for the two-layer *model* is determined in an analogous manner from an initial non-linear flow pattern in the *atmosphere*, the relations between the flow patterns of the two-layer model and the pertinent patterns in the atmosphere remain the same for all later time; in other words, the systems are dynamically similar in this sense.

The above hypothesis involves a tacit assumption that the continuous baroclinic *model* with a rigid top is a satisfactory approximation to the actual *atmosphere*. One obvious difficulty here is the lack of a stratosphere in the continuous baroclinic model (or the two-layer model) that is considered. Otherwise, the properties of the disturbances in the continuous baroclinic model, as they have been discussed by Eady [7], Fjørtoft [9] and Charney [2], agree quite well with typical developments as seen on surface and upper-air charts.

Equations (3) and (4) will therefore be applied, in their linearized form, to the basic current system illustrated in fig. 4. The zonal velocities U and U' are independent of y (latitude), and the Coriolis parameter f will be set equal to $f_0 + \beta y$, where it is to be differentiated with respect to y, but will be considered constant (f_0) when it appears in undifferentiated form. ($\beta = 2\omega R^{-1} \cos \phi_0$, where ω is the angular velocity of the earth, ϕ_0 the central latitude, and R the radius of the earth.) The undisturbed basic-current system must be in hydrostatic and geostrophic equilibrium, so that the following relation (Margules' formula, or thermal wind equation) exists between U', U and the slope dh/dy of the undisturbed interface:

$$U = \epsilon U' - (1 - \epsilon)(g/f_0)\, dh/dy. \tag{10}$$

It is convenient and sufficient for our purposes to restrict our considerations to perturbations which are not a function of latitude. We can then put $\zeta' = \partial v'/\partial x$ and $\zeta = \partial v/\partial x$, where v' and v are the perturbation meridional velocity-components in the upper and lower fluids, respectively. If the perturbation quantities are put proportional to exp $[ik(x - ct)]$, where $k = 2\pi L^{-1}$ is the wave number and c the phase speed of the wave perturbation, the following linearized forms of the potential vorticity equations (3) are readily obtained:

$$[-(U - c)k^2 + \beta]v - \frac{f_0}{D}\left[(U - c)\frac{\partial \delta}{\partial x} + v\frac{dh}{dy}\right] = 0, \tag{11}$$

$$[-(U' - c)k^2 + \beta]v' + \frac{f}{D}\left[(U' - c)\frac{\partial \delta}{\partial x} + v'\frac{dh}{dy}\right] = 0. \tag{12}$$

Here $\delta(x, t)$ is the perturbation of the internal discontinuity surface, and a constant height D has been written in place of the semi-constant heights h and $H - h$. The mean values of h and $H - h$ have then been taken to be equal. This is not only a convenient choice, but seems most consistent with the kinematics of the large-scale disturbances we are trying to imitate.

We may now introduce the hydrostatic and quasi-geostrophic assumptions for the perturbations by using the thermal wind equation to evaluate $\partial\delta/\partial x$ in terms of v' and v. We find

$$(f_0/D)\, \partial\delta/\partial x = (v - \epsilon v')\lambda^2, \tag{13}$$

where λ^2, having the same dimensions as k^2, has been written in place of $f_0^2/[(1 - \epsilon)gD]$. The following two simultaneous equations in v and v' then result when (13) is substituted in (11) and (12):

$$\left[-(U - c)(k^2 + \lambda^2) + \beta - \frac{f_0}{D}\frac{dh}{dy}\right]v + \epsilon\lambda^2(U - c)v' = 0,$$

$$\lambda^2(U' - c)v + \left[-(U' - c)(k^2 + \epsilon\lambda^2) + \beta + \frac{f_0}{D}\frac{dh}{dy}\right]v' = 0.$$

dh/dy can, of course, be eliminated by (10). If v' and v are not to vanish identically, the determinant of their coefficients must vanish. This gives us the frequency equation for the speed of propagation of the small perturbations which satisfy the linearized equations. It is a quadratic equation for c and has a solution of the form $c = [-b \pm (\Delta)^{\frac{1}{2}}]/2a$. If the discriminant Δ is negative, c is complex, which means that such perturbations are proportional to exp $[\pm k(-\Delta)^{\frac{1}{2}}t]$ and will grow or decrease exponentially with time, according to the choice of the plus or minus sign in front of the radical.[5] Before discussion of the properties of the frequency equation, it will be convenient to define two new velocities, $\bar{U} = \frac{1}{2}(U + U')$ and $U^* = \epsilon U' - U$. The non-dimensional parameter α, defined by $\alpha = k^2/\lambda^2$, will also be useful.

When the quadratic equation is solved for $c\ (= c_r \pm c_i)$, we find that the part of the phase velocity which is always real (c_r) has the following value:

$$c_r = \bar{U} - \frac{\beta}{k^2}\frac{2\alpha + (1 + \epsilon)}{2\alpha + 2(1 + \epsilon)} - \frac{(1 - \epsilon)U^*}{2\alpha(\alpha + 1 + \epsilon)}. \tag{14}$$

The part of the wave speed which may be imaginary has the following value:

$$c_i = [2(1 + \epsilon)\alpha(\alpha + 1 + \epsilon)]^{-1} \times \sqrt{\begin{aligned} &U^{*2}\{(1 - \epsilon)^2(1 + \epsilon)^2 + 4\alpha^2[\alpha^2 - (1 + \epsilon)^2]\} + \{2(1 - \epsilon)\bar{U}\alpha(\alpha + 1 + \epsilon) - (1 + \epsilon)^2\beta\lambda^{-2}\}^2 \\ &\quad + 8\epsilon(1 - \epsilon)(1 + \epsilon)\alpha(\alpha + 1 + \epsilon)\beta\bar{U}\lambda^{-2} \\ &\quad + 2(1 - \epsilon)U^*[2\alpha^2 - (1 + \epsilon)^2]\{2\alpha(\alpha + 1 + \epsilon)\bar{U} - (1 + \epsilon)\lambda^{-2}\beta\}\end{aligned}}. \tag{15}$$

It is clear from this last expression that, although a necessary condition for the radicand to be negative is that $U^* \neq 0$ (if we consider $0 < \epsilon < 1$ and $\bar{U} \geqq 0$), the complete stability criterion depends on U^*, $\bar{U}$, β, λ^2, α and ϵ.

The properties of the perturbations on the basic state of the two-layer model will now be compared with the properties of similar perturbations in a continuous baroclinic atmosphere with a rigid top, as they have been studied by Eady [7] and Fjørtoft [9].

[5] It is not necessary to make the quasi-geostrophic assumption (13). Since the perturbations are independent of latitude, $\partial u/\partial x = \text{div}\, \mathbf{v} = -D^{-1}[(U - c)\, \partial\delta/\partial x + v\, dh/dy]$. If the x equation of motion is linearized, $(U - c)\, \partial u/\partial x = -g\, \partial z/\partial x + fv$. These can be combined into an equation of the form $lv + m\, \partial\delta/\partial x = \partial z/\partial x$, and a similar equation, $l'v' + m'\, \partial\delta/\partial x = \epsilon\, \partial z'/\partial x$, can be obtained for the upper fluid. (l, m, l' and m' are constants.) The hydrostatic equation, $z = \epsilon z' + (1 - \epsilon)h +$ const, can then be introduced to give an equation of the form $\partial\delta/\partial x = Mv + Nv'$. This can be used in place of the quasi-geostrophic equation (13) and leads finally to a *quartic* frequency equation instead of the *quadratic* equation for c obtained above. However, since the non-linear model uses the geostrophic assumption, it was not considered necessary or even advisable to use this refinement in studying the perturbation behavior.

Eady's results are more convenient for this purpose. He discusses in most detail the case where the effect of the variability of the Coriolis parameter is neglected ($\beta = 0$), and his results for this case (with perturbations independent of y) may be summarized as follows:

1. Waves of a length shorter than a critical length are neutral (c is real), while waves longer than this critical length have complex values of c and are therefore either amplified or damped. This critical wavelength is a function only of the Coriolis parameter f, the total depth ($2D$), and the vertical stability $[\partial(\ln \vartheta)/\partial z]$. In terms of a critical wave number k_c, he finds

$$k_c^2 = (1.1997f)^2/[gD^2\,\partial(\ln \vartheta)/\partial z]. \qquad (16)$$

2. There is a certain wavelength ($k < k_c$) which is most unstable. If $v_{i\,\max}$ ($= k|c_i|$) is the "time of flight" for this most unstable wave,

$$v_{i\,\max}^2 = (0.31f\,dU/dz)^2/[g\,\partial(\ln \vartheta)/\partial z]. \qquad (17)$$

3. The real part of the phase velocity has the value

$$c_r = \tfrac{1}{2}(U_t + U_0), \qquad (18)$$

where U_t and U_0 are the zonal velocities at the top and bottom of the continuous, baroclinic basic current, respectively.

We must now see whether the perturbations in the two-layer model have the same properties when $\beta = 0$. A study of the radicand in (15) immediately shows that the stability criterion then depends on $\bar{U}$ and U^* in addition to the vertical stability, Coriolis parameter, depth and wave number (the latter quantities being contained in α). It is convenient to divide the radicand in (15) by $\bar{U}^2$ and set the resulting expression equal to zero, to investigate the form of the criterion.

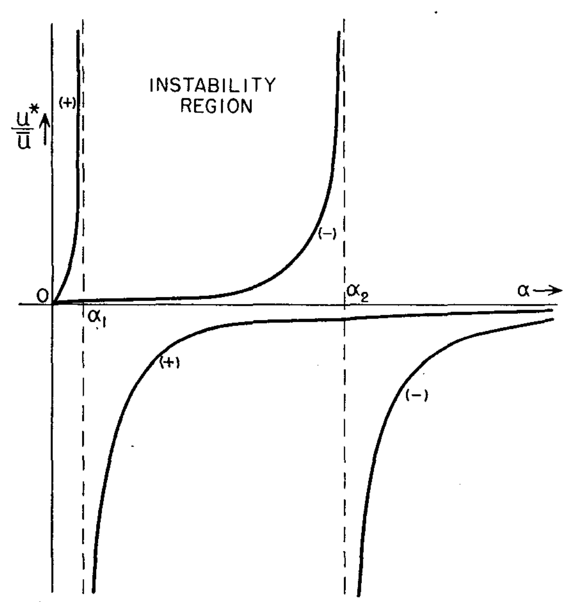

FIG. 5. Schematic illustration of stability criterion for two-layer linearized model with $\beta = 0$. Heavy lines show two roots, (+) and (−), of (19) as functions of α (proportional to square of wave number for fixed values of Coriolis parameter, density ratio, gravity and depth). Dashed lines are asymptotic values for large values of $U^*/\bar{U}$ ($U^*/\bar{U}$ being proportional to the ratio of momentum shear to mean zonal wind in basic current).

For $\beta = 0$, this yields

$$(U^*/\bar{U})^2\{(1-\epsilon)^2(1+\epsilon)^2+4\alpha^2[\alpha^2-(1+\epsilon)^2]\}$$
$$+(U^*/\bar{U})\{4(1-\epsilon)\alpha[2\alpha^2-(1+\epsilon)^2][\alpha+1+\epsilon]\}$$
$$+\{2(1-\epsilon)\alpha[\alpha+1+\epsilon]\}^2=0 \qquad (19)$$

as the critical condition. This equation is most conveniently studied as a quadratic equation in $(U^*/\bar{U})$ whose coefficients are functions of α and ϵ. The two roots of this equation are plotted schematically in fig. 5, and the region of instability indicated thereon. (The curves for $U^*/\bar{U} < 0$ are included for completeness but are of no particular interest here.) The dashed vertical asymptotes, $\alpha = \alpha_1$ and $\alpha = \alpha_2$, are the two positive values of α for which the coefficient of $(U^*/\bar{U})^2$ in (19) vanishes, and are given approximately by $\alpha_1 = 0.5(1 - \epsilon)$ and $\alpha_2 = (1 + \epsilon)$. This figure brings out the fact that the stability criterion for the two-layer model with $\beta = 0$ is a function of $(U^*/\bar{U})$ as well as α. Eady's criterion (16), on the other hand, involved only quantities similar to those contained in our α, and his criterion would accordingly be represented by a single vertical line on fig. 5. Furthermore, the stability region for small α ($= k^2/\lambda^2$) is completely absent in Eady's model. These apparent discrepancies must be resolved before we can assume that the two-layer model will be a satisfactory approximation to the continuous atmosphere, even in the linear case.

Fortunately, the source of these discrepancies is easily determined: In setting up the equations which apply to their baroclinic models, Eady and Fjørtoft use the usual simplified form of the thermal wind equation which results when the gradient of potential temperature along an isobaric surface is replaced by the horizontal gradient of potential temperature. Equations (10) and (13), on the other hand, are exact formulations of the thermal wind equation. If $\epsilon U'$ and $\epsilon v'$ are replaced by U' and v' in (10) and (13), respectively, we introduce essentially the same simplification as that used by Eady and Fjørtoft. The value of c obtained when this is done is much simpler than that given by (14) and (15). We find, ultimately,

$$c = \bar{U} - (\beta/k^2)[(1+\alpha)/(2+\alpha)]$$
$$\times[2\alpha(2+\alpha)]^{-1}[(U'-U)^2\alpha^2(\alpha^2-4)+4\beta^2/\lambda^4]^{\frac{1}{2}}. \quad (20)$$

This can be obtained directly from (14) and (15) by replacing ϵ by 1 where it appears explicitly in those equations, and by replacing $U^* = \epsilon U' - U$ with $U' - U$.[6] For $\beta = 0$, the stability criterion is now $\alpha \gtrless 2$ ($\alpha < 2$ corresponding to amplified or damped waves) and we then have, from the definition of α,

$$k_c^2 = 2\lambda^2 = 2f^2/[g(1 - \epsilon)D]. \quad \text{(2-layer)} \quad (21)$$

The value of α which corresponds to maximum instability is easily determined and turns out to be

[6] When the negative root is taken, this expression "reduces to Rossby's formula" for $U' = U$.

$\alpha = 2[(2)^{\frac{1}{2}} - 1]$. For this value of α, the time of flight is given by

$$v_{i\,\max}^2 = 0.086\lambda^2(U' - U)^2. \quad \text{(2-layer)} \quad (22)$$

Finally, we have the real part of the phase velocity (still with $\beta = 0$):

$$c_r = \bar{U}. \quad \text{(2-layer)} \quad (23)$$

We now require that the linearized two-layer model have the same values of k_c^2, $v_{i\,\max}^2$ and c_r as the continuous model of Eady. Equating (21) with (16), (22) with (17), (23) with (18), and taking the total depth ($2D$) to be the same, we find

$$\begin{aligned} (1 - \epsilon) &= 0.695(\vartheta_t - \vartheta_0)/\bar{\vartheta}, \\ U' - U &= 0.626(U_t - U_0), \\ U' + U &= U_t + U_0, \end{aligned} \quad (24)$$

where the subscripts t and 0 refer to values at the top and bottom of the continuous model (which we will interpret as the tropopause level and the surface of the earth), respectively, and $\bar{\vartheta}$ is equal to $\frac{1}{2}(\vartheta_t + \vartheta_0)$. When the two-layer model in fig. 4 has values of $(1 - \epsilon)$, U' and U determined according to (24), a superimposed perturbation of an arbitrary wavelength will behave quite similarly to a perturbation of the same wavelength superimposed on a continuous, baroclinic zonal current with the same depth ($2D$) and a zonal wind and vertical stability distribution given by the quantities U_t, U_0 and $(\vartheta_t - \vartheta_0)/\bar{\vartheta}$. Strictly speaking, we have determined this equivalence only when the approximate form of the thermal wind equation is valid, but there seems to be no reason why this approximation should destroy the equivalence when the exact form of the thermal wind equation is used in the two-layer model. Unfortunately, there does not appear to be any study of these perturbations in a continuous baroclinic current with rigid top in which the exact form of the thermal wind equation is used.[7]

Having determined the conditions for dynamic similarity without reference to the effect of a variable Coriolis parameter, we must compare the two-layer and continuous models in this respect. Eady does not discuss in any detail the effect of β on the stability criterion, so we will compare our results with those of Fjørtoft [9]. Fig. 6 contains a comparison of the stability criteria for long waves for our two-layer model, Fjørtoft's continuous incompressible model and Charney's continuous compressible model [2]. All the curves are based on a normal value of the vertical stability ($\partial\vartheta/\partial z = 3.5$C/km), a tropospheric depth of about 10 km, and values of f and β appropriate to 45°N. The curve for the two-layer model was obtained by setting the radicand in (20) equal to zero, and using (24) to interpret $(U' - U)$ and $(1 - \epsilon)$ in terms of dU/dz and $\partial\vartheta/\partial z$ for the continuous models. The agreement between Fjørtoft's curve and that for the two-layer model suggests that (24) is approximately correct also for the case when $\beta \neq 0$, and that the use of (24) to obtain dynamic similarity with respect to gravity and Coriolis forces will not interfere with the requirements for dynamic similarity with respect to the effects of β.

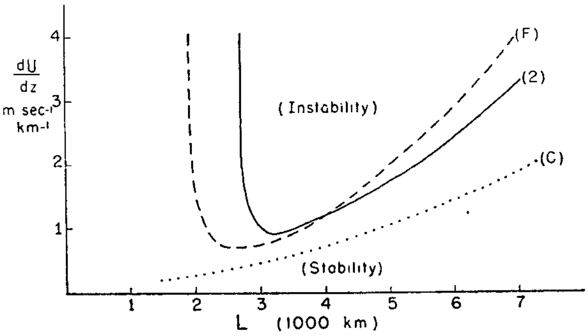

FIG. 6. Stability criteria for various quasi-geostrophic baroclinic models with $\beta \neq 0$. (C): Charney's criterion; (F): Fjørtoft's criterion; (2): two-layer model with rigid top.

The equivalence relations (24), in combination with the thermal wind equation, show that

$$dh/dy \approx 0.9(\partial z/\partial y)_\vartheta, \quad (25)$$

where $(\partial z/\partial y)_\vartheta$ is the mean meridional slope of the isentropic surfaces in the troposphere. It is therefore clear that, when the relations (24) are used, the interface h does not correspond to the conventional concept of a "polar front," but is the dynamic equivalent of the *mean* solenoidal field in the troposphere.

We have established (24) as the equations which define a two-layer model that will behave toward small quasi-geostrophic perturbations in a manner similar to the behavior of a continuous model toward perturbations of the same wavelengths. We now make the hypothesis that the same relations must hold for similarity in a non-linear problem. If z_t and z_0 are the heights of isobaric surfaces near the tropopause level and the surface of the earth, respectively, this implies that

$$\begin{aligned} (1 - \epsilon) &= 0.695(\vartheta_t - \vartheta_0)/\bar{\vartheta}, \\ z' - z &= 0.626(z_t - z_0), \\ z' + z &= z_t + z_0, \end{aligned} \quad (26)$$

are the relations we must use to evaluate the terms in (7) and (5). (We are interested only in the time and space derivatives of z and z', so any additive constants are arbitrary.) In practice, $(\vartheta_t - \vartheta_0)/\bar{\vartheta}$ varies some-

[7] The remarkable structural similarity between (21) and (16), (22) and (17), and (23) and (18) suggests that a more exact analysis of the quasi-geostrophic perturbation problem for the continuous case with $\beta = 0$, using the exact thermal wind equation, might show a dependence of the stability criterion on the relative values of U^* and $\bar{U}$, in analogy with the results contained in (14) and (15). It should also be remarked that, although the two-layer model can evidently be made dynamically similar to a continuous baroclinic model as far as large-scale quasi-horizontal flow patterns are concerned, this, of course, may not be possible for other phenomena, *e.g.*, "Helmholtz" waves.

what, so we must take an average value that is appropriate for the region in which we are interested.

The distribution of h will be determined from (4), after z and z' have been computed from (26). The constant in (4) is relevant, however, since h is used to determine the potential vorticity. We will assume that this constant is chosen so that the mean value of h is equal to half the mean value of the height of the upper isobaric surface, *i.e.*,

$$\iint h\,dx\,dy = \tfrac{1}{2}\iint z_t\,dx\,dy. \qquad (27)$$

This equation defines the constant in (4), so that h is completely determined.

Finally, it is of some interest to compare the field of vertical motion in the two-layer model with that in the atmosphere. If we assume that the latter vertical motions can be represented by $w = W(x, y, t) \sin(\pi z/H)$, so that W is the (maximum) vertical velocity in the middle of the troposphere ($z = \frac{1}{2}H$), the vorticity equations at $z = 0$ and $z = H$ can be used to evaluate $\partial w/\partial z = -\operatorname{div} \boldsymbol{v}$ in terms of the individual changes of vorticity at those levels in the troposphere. These latter quantities, in turn, can be expressed in terms of the vorticity changes in the two-layer model by means of (26). One finds, finally, that

$$W \approx 2(0.626\pi)^{-1}\,dh/dt \approx dh/dt. \qquad (28)$$

This means that by means of (26) the two-layer model has been chosen so that the values of the divergence in the lower and upper layers are, respectively, equal to the (vertical) mean values of the divergence within the lower and upper half of the troposphere—a result which increases, perhaps, the credibility of (26).

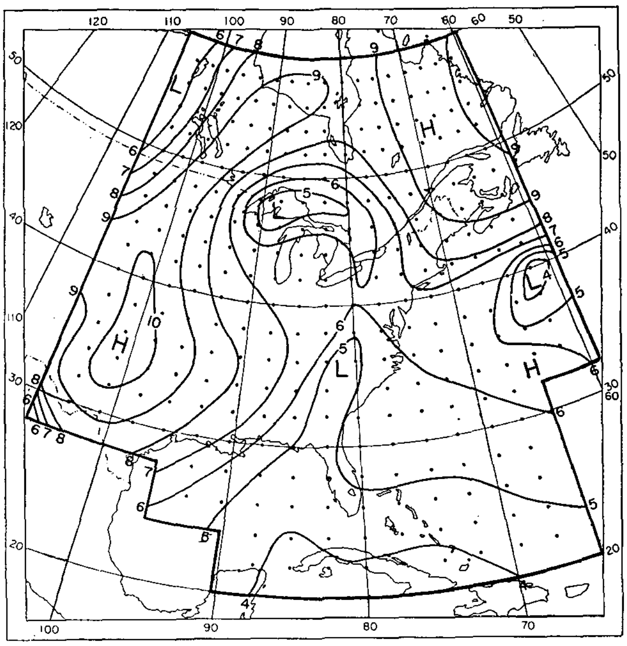

FIG. 7. 1000-mb chart for 1500 GCT 24 November 1950. Sea-level pressures and 3-hr tendencies at 1830 GCT were used in combination with 1000-mb heights reported on radiosonde observations at 1500 GCT to obtain chart. Isolines are height contours in hundreds of feet above sea level.

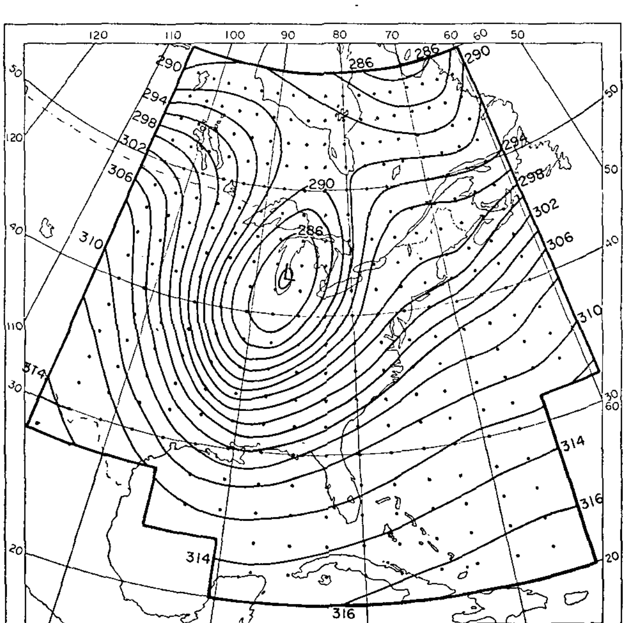

FIG. 8. 300-mb chart for 1500 GCT 24 November 1950. Isolines are height contours at intervals of 200 ft.

4. Application to a synoptic example

A preliminary test of the model has been made by calculating the field of vertical motion and "sea-level" pressure tendencies over eastern North America during an early stage of the development of the severe storm of 25 November 1950 in that region.[8] The initial

[8] The history of this storm has been discussed by Smith [17].

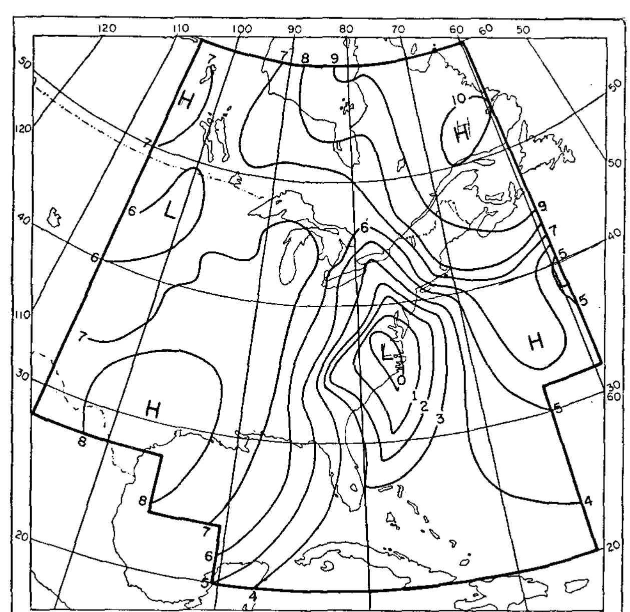

FIG. 9. 1000-mb chart for 0630 GCT 25 November 1950.

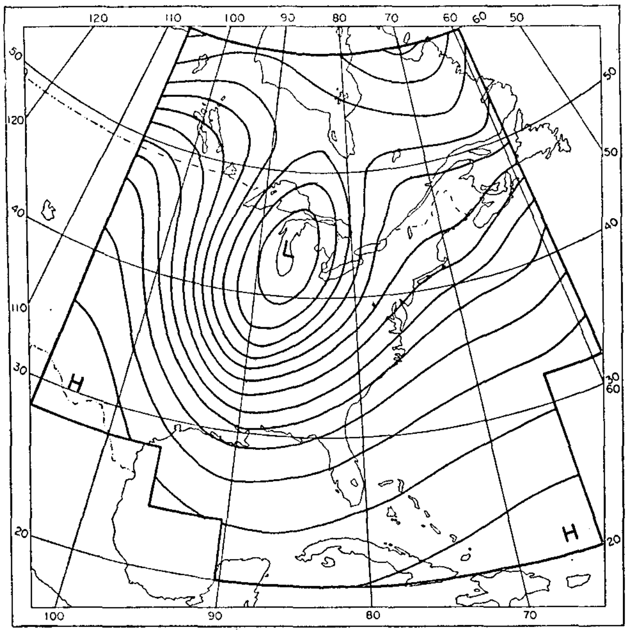

FIG. 10. Isolines of z' at 200-ft intervals for 1500 GCT 24 November 1950.

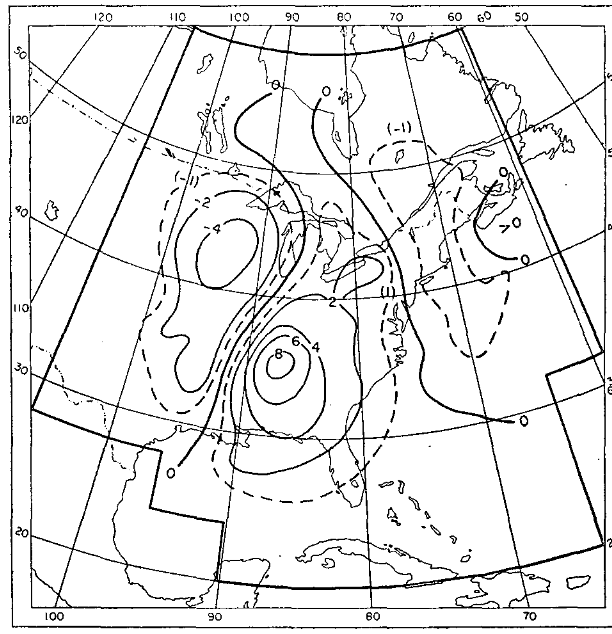

FIG. 12. $\partial h/\partial t$ (cm/sec) obtained from solution of (7) for 1500 GCT 24 November 1951.

situation chosen for study was that at 1500 GCT 24 November 1950. Figs. 7 and 8 are the 1000- and 300-mb charts, respectively, for this time and were used to give the values of z_0 and z_t in (26). The ensuing surface development is shown by fig. 9, the 1000-mb chart for 0630 GCT 25 November, 15 hr later than the chart in fig. 7. The intensity of the vortex on the 300-mb chart in fig. 8 shows that this example is a rather severe test of the model insofar as the quasi-geostrophic assumption is concerned, since the strong curvature of the contours must correspond to an appreciable deviation of the actual wind from the geostrophic value. Furthermore, the horizontal variation of the mean vertical stability between the 1000- and 300-mb levels [as expressed by $(\vartheta_{300} - \vartheta_{1000})/\bar{\vartheta}$] was more pronounced than normal, the vertical stability in the center of the vortex being quite large.

The heavy line on the charts outlines the area over which computations were made, and the grid that was used to solve the finite-difference analogues of (7) and (5) is also included in figs. 7 and 8. For simplicity,

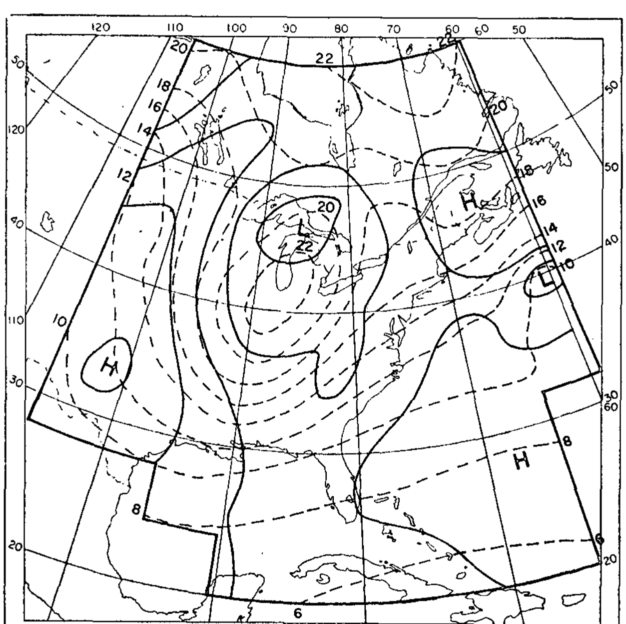

FIG. 11. Isolines of z (solid lines) at 200-ft intervals and h (dashed lines) at 2000-ft intervals for 1500 GCT 24 November 1950.

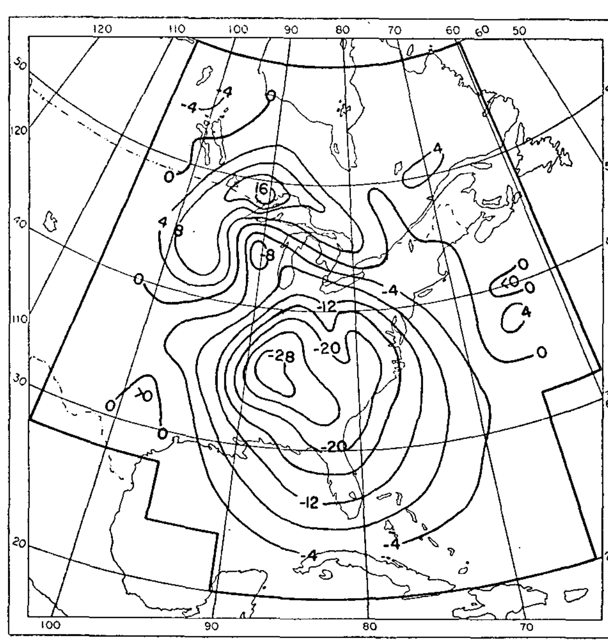

FIG. 13. Pressure-height tendency, $\partial z/\partial t$, for lower fluid in two-layer model, obtained from solution of (5) for 1500 GCT. Units are 10^{-2} cm/sec.

the Coriolis parameter was defined as $f_{40} + \beta_{40}y$, where f_{40} and β_{40} are the values of f and β at 40°N. The convergence of the meridians was ignored, by assuming that the grid points (figs. 7 and 8) formed squares whose sides were of a length equal to two degrees of latitude. Some distortion is thereby introduced, but, as shown by Haurwitz [12], this is not a very great error.

A value of ϵ was first obtained by averaging the value of $2(\vartheta_{300} - \vartheta_{1000})/(\vartheta_{300} + \vartheta_{1000})$ for all aerological soundings at this time in the region, and setting this mean value equal to $(0.695)^{-1}(1 - \epsilon)$, according to (26). This gave values of 0.889 for ϵ and 0.111 for $(1 - \epsilon)$. The distributions of z', z and h were then determined, as outlined at the end of the last section, and are shown in figs. 10 and 11.

Next, the coefficients in (7) were computed, and (7) was solved by hand computation, using a modification of Southwell's relaxation method [18] and assuming $\partial h/\partial t$ to be zero on the boundaries. (The boundaries for this part of the computation were one grid point inside the heavy outline on the charts, since the computation of the various advection terms, *e.g.*, $\mathbf{v}\cdot\nabla\eta$, requires pressure-height data from two points on each side of the point for which it is being computed.) The field of $\partial h/\partial t$ obtained by this process is shown in fig. 12.

After computation of $\partial h/\partial t$, (5) could be solved for $\partial z/\partial t$. This was also done by relaxation, with $\partial z/\partial t \doteq 0$ on the same boundaries as for $\partial h/\partial t$.[9] Fig. 13 contains the results of this computation.

Sea-level pressure tendencies were then computed by the following procedure. First, the last two equations in (26) were differentiated locally with respect to time. $\partial z'/\partial t$ was then eliminated from each of them by using (4), also differentiated with respect to time:

$$\frac{\partial z'}{\partial t} = \frac{1}{\epsilon}\left[\frac{\partial z}{\partial t} - (1 - \epsilon)\frac{\partial h}{\partial t}\right].$$

The resulting two equations were then solved for $\partial z_0/\partial t$:

$$\frac{\partial z_0}{\partial t} = \frac{1}{1.252\epsilon}\left[(1.626\epsilon - 0.374)\frac{\partial z}{\partial t} + 0.374(1 - \epsilon)\frac{\partial h}{\partial t}\right].$$

For comparison with observed sea-level pressure tendencies, the predicted 1000-mb height tendency $\partial z_0/\partial t$ was then converted into an equivalent pressure tendency, by multiplication with $\rho_0 g$. ($\rho_0 = 1.25 \times 10^{-3}$ g/cm³.) If $\partial p_0/\partial t$ is in units of mb (6 hr)$^{-1}$, and $\partial z/\partial t$ and $\partial h/\partial t$ are in the same units as they are in figs. 13 and 12 (10^{-2} cm/sec and cm/sec, respectively), this results in the following equation for the pressure tendency:

$$\left(\frac{\partial p_0}{\partial t}\right) = 0.255\left(\frac{\partial z}{\partial t}\right) + 0.984\left(\frac{\partial h}{\partial t}\right). \qquad (29)$$

It is clear from this equation, and the charts of $\partial z/\partial t$ and $\partial h/\partial t$, that the sea-level pressure tendency will usually be the small sum of two larger quantities with opposing signs. $\partial z/\partial t$ can be thought of as the pressure-height tendency of an isobaric surface in the middle troposphere ($\approx$ 800 mb), while $\partial h/\partial t$ can be considered as approximately proportional to the negative of the

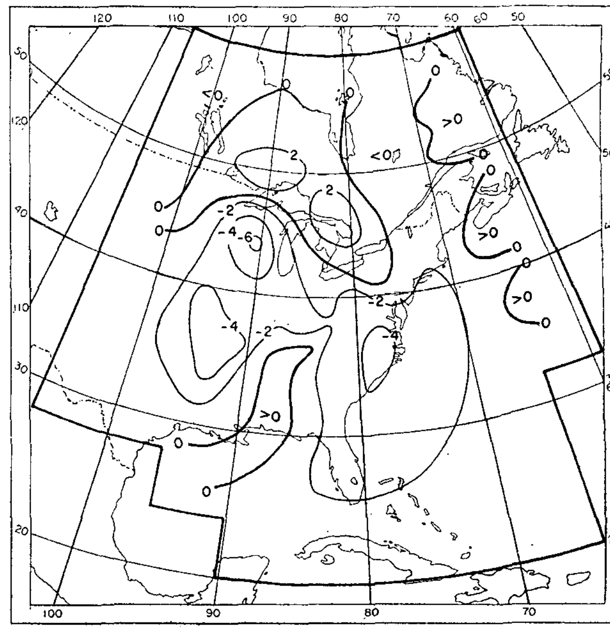

FIG. 14. Sea-level pressure tendencies at 1500 GCT in mb (6 hr)$^{-1}$, calculated from two-layer model with rigid top.

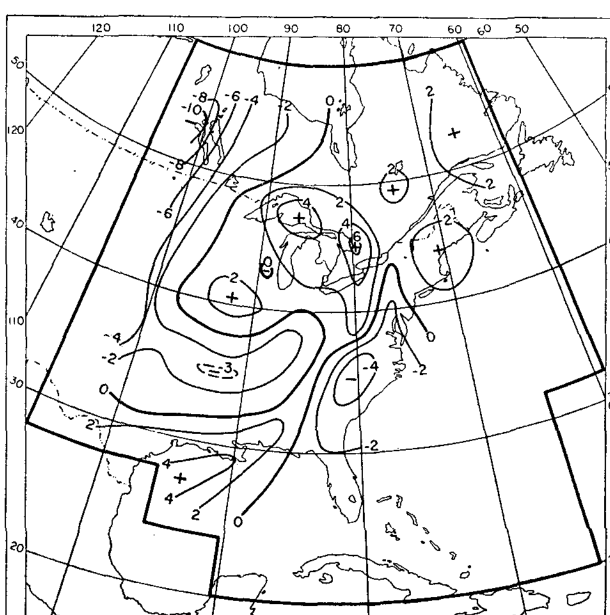

FIG. 15. Observed sea-level pressure changes in mb from 1230–1830 GCT 24 November 1950, corrected for normal diurnal change.

[9] These artificial boundary conditions make the computations less meaningful at grid points close to those boundaries, and therefore no great significance should be assigned to the edges of the areas covered by isolines on figs. 12, 13, 14 and 16.

tendency of the thickness between 1000 and 300 mb, $\partial\Delta/\partial t$, say. It is common experience that large positive values of $\partial\Delta/\partial t$, for example (corresponding to large negative values of $\partial h/\partial t$), are usually associated with large pressure rises in the middle and upper troposphere, so that the pressure tendency at the surface remains small. Equation (29), together with figs. 12 and 13, shows that the same state of affairs is true in our model.

The tendencies computed from (29) are shown in fig. 14, and the observed pressure changes between 1230 and 1830 GCT, corrected for normal diurnal change, are shown in fig. 15. A comparison of these figures shows that the principal feature of the errors in the computed field is a region of too great pressure-fall, centered west of the Great Lakes.

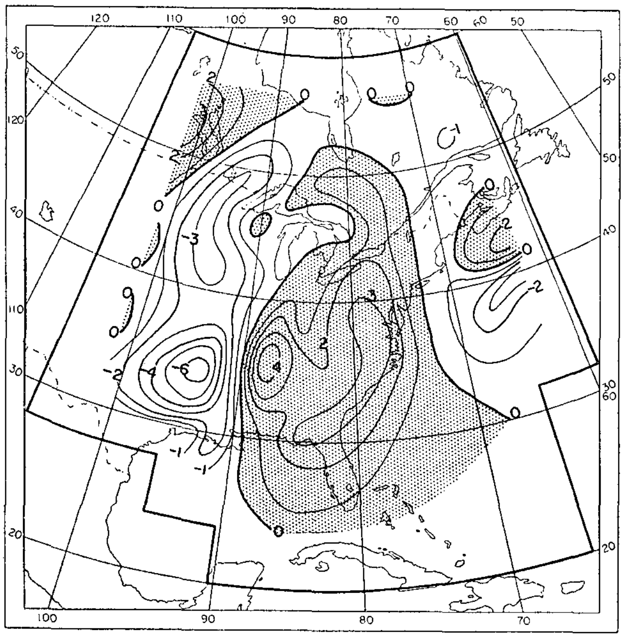

FIG. 16. Vertical velocity w_h of lower fluid at interface in model (cm/sec). Areas of positive w_h (rising motion) shaded.

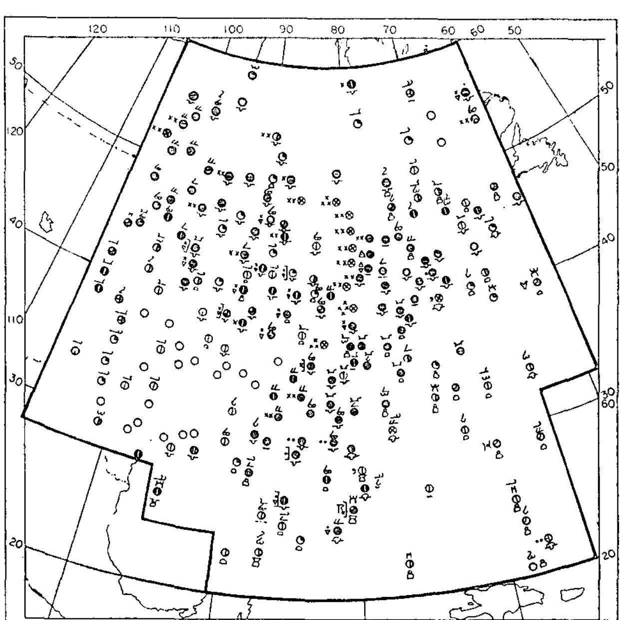

FIG. 17. Precipitation and clouds at 1830 GCT 24 November 1950.

The field of w_h was next computed from the kinematic boundary condition at h:

$$w_h = dh/dt = \partial h/\partial t + \boldsymbol{v}\cdot\nabla h,$$

($\boldsymbol{v}\cdot\nabla h$ being evaluated geostrophically). According to (28), w_h corresponds to the vertical motion in the center of the troposphere. Fig. 16 contains the results of this computation while, for comparison, fig. 17 shows the clouds and weather observed at 1830 GCT on the 24th. Although the general pattern of w_h agrees fairly well with the observed hydrometeor distribution, it appears that the center of +4 cm/sec in the field of w_h at 35°N and 88°W is probably an overestimate. (The amount of descending motion west of the Great Lakes may also be somewhat overestimated.)

The source of the errors in the computed surface pressure tendencies and vertical motion patterns may be grouped roughly into three categories:

1. The restricted degrees of freedom of the model with respect to the vertical coordinate;
2. The quasi-geostrophic assumption;
3. Other assumptions contained in the model.

Under (1), one would include such items as the lack of anything in the model which might correspond to the stratosphere and the uniform value of the vertical stability that is inherent in the model. (Horizontal variations of the stability imply more complicated distributions of *vertical* wind shear.) Group (3) would include such assumptions as the neglect of viscosity, turbulence, heat exchange, compressibility, sphericity of the earth, and the use of a finite-difference network to solve the differential equations. It is, of course, impossible to conclude from this single example which of the various assumptions may be most instrumental in causing the errors. Some speculation may not be out of place, however, as long as it is recognized that the speculation is based on this single case.

First of all, it does not seem reasonable that the rigid top we have imposed on our model can represent completely the influence of the stratosphere on the motions of the troposphere. A seemingly more realistic model may be formulated by superimposing an infinitely deep fluid of still smaller density ρ'' above the fluid layer ρ'. The infinite depth and barotropy of this upper fluid allow us to assume that there are no horizontal motions in the new layer and that the pressure in this upper layer is a function only of height. The interface height H must now be a function of x, y and t, since the pressure-height z' in the middle layer is now produced hydrostatically. In fact, if $\epsilon' = \rho''/\rho'$, we have that $z' = (1 - \epsilon')H + \text{const}$. The only modification this introduces into the vorticity equations (5) and (6) is to add a term $-f\eta'[g(H-h)]^{-1}\,\partial H/\partial t$ to the left side of (6). ($v'\cdot\nabla H$ is zero if v' is geostrophic.) However, if we try to obtain a differential equation containing only $\partial h/\partial t$ or only $\partial H/\partial t$ from (4), (5) and the modified form of (6), we find that it is necessary to apply the operator ∇^2 to either (5) or (6) before this can be accomplished. As a result, the equation contains either $\nabla^4(\partial h/\partial t)$ or $\nabla^4(\partial H/\partial t)$ as the highest order term.

When the ratio ρ''/ρ' is appreciably smaller than ρ'/ρ, corresponding to greater gravitational stability at H than at h, one can expect that $\partial H/\partial t \ll \partial h/\partial t$. This suggests that, under these conditions, the divergence in the middle layer may, to a first approximation, be calculated without taking $\partial H/\partial t$ into account, just as was done with the rigid-top model. One might then take the value of $\partial z'/\partial t$ calculated from the rigid-top model, and use it to evaluate the term $-f\eta'[g(H-h)]^{-1}\,\partial H/\partial t$ in the modified form of (6) (by using the hydrostatic relation, $\partial H/\partial t = (1-\epsilon')^{-1}\,\partial z'/\partial t$). It is then possible to solve (5) and (6) again to yield a second approximation to $\partial z'/\partial t$, in which some of the effect of $\partial H/\partial t$ has been taken into account. This procedure may be repeated, and it can be shown that this iteration will converge to the correct value of $\partial z'/\partial t$ (the value obtained when the complete biharmonic equation is used), if the value of ϵ' is sufficiently small and the "wavelength" of the disturbances is not too large. The first step in this iterative process was carried out for the synoptic situation studied above. If $\Delta(\partial p_0/\partial t)$, $\Delta(\partial h/\partial t)$, and $\Delta(\partial z/\partial t)$ are the changes in the respective quantities produced by this first step in the successive approximation, one finds that

$$(1-\epsilon')\,\Delta(\partial p_0/\partial t) = (1-\epsilon')(0.255)\,\Delta(\partial z/\partial t) + (1-\epsilon')(0.984)\,\Delta(\partial h/\partial t),$$

where the right (and left) sides of the equation are independent of the choice of $(1-\epsilon')$. [In other words, $\Delta(\partial z/\partial t)$ and $\Delta(\partial h/\partial t)$ are themselves inversely proportional to $1-\epsilon'$.] In the application to the example of 24 November, it turned out that the right side of this equation was everywhere less than 0.15 mb $(6\text{ hr})^{-1}$, so that a realistic value of $(1-\epsilon')$, say 0.2 or 0.3, would not produce changes in the surface pressure tendency sufficient to account for the observed errors. Although it is conceivable that the inclusion of more of the successive approximations might increase $\Delta(\partial p_0/\partial t)$ somewhat, this approach did not seem promising and was not carried any further. This, of course, does not mean that the stratosphere plays no important role in tropospheric developments, but only that the addition of an inert third layer to the two-layer model does not seem to affect the motion significantly, when the added gravitational stability represented by this modification has a reasonable value.

Perhaps one of the most important sources of error may be in the application of the quasi-geostrophic assumption to a situation characterized by large centripetal accelerations in the upper troposphere (fig. 8). The replacement of the quasi-geostrophic assumption by a similar statement based on the assumption of gradient wind balance is quite complicated. It seems reasonable, however, that, since most of the flow pattern in our example is characterized by cyclonic curvature, the geostrophic evaluation of the advection term has led to an overestimate of the magnitude of these terms, and that this overestimate would be particularly important in the upper layer, *i.e.*, in the term $v'\cdot\nabla\eta'$. This would be most significant in the region just downstream and upstream from the vorticity maximum in the upper fluid (located at 38°N, 90°W, on fig. 10). The maximum observed absolute value of $v'\cdot\nabla\eta'$ occurred just east and south of this point, and was negative. It therefore contributed greatly to the nearby maximum of $\partial h/\partial t$ on fig. 12. [See (7) and (9).] A reduction of $\partial h/\partial t$ in this vicinity would give a better picture of w_h, by reducing the +4 cm/sec maximum of w_h in fig. 16, which does not fit in very well there with the observed hydrometeors. It is also possible that the fictitious pressure fall, west of Lake Michigan on fig. 14, might be diminished or removed if the advective terms were reduced in magnitude. A comparison of figs. 12, 13 and 14, with (30) in mind, shows that this error cannot very well be ascribed entirely to too large negative values of $\partial h/\partial t$ arising from an overestimate of $v'\cdot\nabla\eta'$ upstream from the vorticity maximum in the upper fluid. On the other hand, the values of $\partial z/\partial t$ in this vicinity will be made less negative if the negative values of $v\cdot\nabla\eta$ were reduced because of the cyclonic curvature (fig. 11). It seems plausible, therefore, that a better prediction of both the surface pressure tendency and vertical motion could be made if some means were found to modify the quasi-geostrophic assumption so as to take account of the effect of large normal accelerations on the wind distribution.

One extension of the model which would be rather straightforward would be to incorporate the effect of orography by adding a term $f\eta[g(h-z_g)]^{-1}\,v\cdot\nabla z_g$ to the vorticity equation for the lower fluid, where z_g is the elevation of the surface of the earth (suitably "smoothed," of course) above sea-level. Recent work on the effect of orography on the general circulation [1; 5] indicates that this may be an important effect in the vicinity of large mountain complexes and massifs.

It should also be possible to obtain an estimate of part of the non-geostrophic component of the wind by setting this component of the wind equal to the gradient of a velocity potential ϕ such that the divergence of this wind component, given by $\nabla^2\phi$, is equal to the divergence computed from the quasi-geostrophic assumption. Once it is determined, by solving the resulting Poisson equation for ϕ, this non-geostrophic wind could then be incorporated into the various advection terms as a second approximation.

5. Summary, conclusions and suggestions for further work

It is appropriate at this point to summarize briefly the material of the preceding sections. First of all, aerological evidence shows that one of the most important features of the large-scale disturbances of extratropical latitudes is the vertical distribution of horizontal velocity divergence. Normally this quantity is of opposite sign in the lower and upper halves of the troposphere, so that the vertical mean of the horizontal divergence is much smaller than the value at most levels in the troposphere. This feature can be reproduced in a simple two-layer model, whose motions are restricted to those which are hydrostatic, quasi-horizontal and quasi-geostrophic. With these assumptions, the time rates-of-change of the flow patterns and mass distribution in the model are determined by partial differential equations, similar to the familiar Poisson differential equation, in which the tendencies are the only unknown quantities. These equations are adaptable to numerical forecasting.

To obtain a suitable set of relations between the flow patterns of the model and those of the atmosphere, so that a forecast for the model will lead to a forecast for the atmosphere, the behavior of the two-layer model was compared with the behavior of a continuous baroclinic model for an especially simple case—that of small perturbations superimposed on a baroclinic zonal current. The requirement that the perturbations in the two models have the same phase speed and instability properties leads to relations which, when hypothesized to apply also for non-linear flow patterns, determine the pressure-height fields and gravitational stability of the two-layer model from the pressure-height patterns (at the surface and tropopause levels) and mean gravitational stability of the troposphere.

The application of these results to an actual synoptic example showed that many of the important features of the vertical-motion and sea-level pressure-tendency patterns can be inferred from the model. However, the computed pattern showed some significant deviations from the actual pattern, so that, if this one example can be considered as representative, some further modifications of the model are necessary before it can be considered as a valuable forecast tool.

Two likely modifications might be the incorporation of a third inert layer to represent the stratosphere and the correction of the quasi-geostrophic assumption so as to take into account the effect on the wind field of large normal accelerations. However, it would seem most profitable to apply the model to several other synoptic situations before attempting any revision. It would be especially instructive to compare cases where the flow is strongly curved, as in the example discussed above, with cases of less strongly curved flow.

Perhaps one of the most important uses of the model may be its use in determining the subsequent behavior of idealized initial states. The adaptability of the model to numerical forecasting techniques suggests that it may be especially valuable as a tool in the study of non-linear flow patterns—these being exceedingly difficult to treat with conventional mathematical tools. An example of such a problem is the motion of baroclinic cold anticyclones, as discussed by Rossby [15]. The model may also be valuable in studying some linear problems which are intractable. Such a problem is the study of the behavior of small disturbances superimposed on a basic, baroclinic zonal current which varies with latitude. Although the behavior of perturbations on an arbitrary barotropic non-divergent zonal current has been studied by Kuo [13] and Fjørtoft [9], and the behavior of perturbations on a basic baroclinic zonal current which is independent of latitude has been studied by Charney [2], Eady [7], Fjørtoft [9] and others, the combined problem has not as yet succumbed to conventional perturbation methods.

Of course, the general quasi-geostrophic baroclinic model proposed by Charney can also be applied to this type of problem, and is indeed capable of including more complicated dynamic effects than the simpler two-layer model; but this feature is not always an advantage, since it is then more difficult, in a given situation, to separate the important from the unimportant dynamic effects. As an example of this, one may cite the recent forecasts made by Charney and others with the barotropic model [6]. These forecasts correctly predicted certain cases of "development" at the 500-mb level. If these forecasts had been made instead from the complete equations of motion, it probably would not have been at all clear that these developments were essentially barotropic and quasi-geostrophic in nature. It seems likely, therefore, that the two-layer quasi-geostrophic model has a proper place in the scheme envisaged by Charney, where the problem of *understanding* the dynamics of the atmosphere is most logically attacked by studying a hierarchy of models, beginning with the most simple and gradually progressing to more and more complicated models.

Acknowledgments.—The writer wishes to acknowledge the benefit he has received from many stimulating discussions on this general subject with Drs. C.-G. Rossby, E. Palmén, J. Charney, R. Fjørtoft, and particularly with Dr. G. W. Platzman of the University of Chicago.

REFERENCES

1. Bolin, B., 1951: On the influence of the earth's orography on the general character of the westerlies. *Tellus*, **2**, 184–195.
2. Charney, J., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **5**, 135–162.
3. ——, 1948: On the scale of atmospheric motions. *Geofys. Publ.*, **17**, No. 2, 17 pp.
4. ——, 1949: On a physical basis for numerical prediction of large-scale motions in the atmosphere. *J. Meteor.*, **6**, 371–385.
5. Charney, J., and A. Eliassen, 1949: A numerical method for predicting the perturbations of the middle latitudes. *Tellus*, **1**, No. 2, 38–54.
6. Charney, J., and others, 1950: Numerical integration of the barotropic vorticity equation. *Tellus*, **2**, 237–254.
7. Eady, E. T., 1949: Long waves and cyclone waves. *Tellus*, **1**, No. 3, 33–52.
8. Eliassen, A., 1949: The quasi-static equations of motion with pressure as independent variable. *Geofys. Publ.*, **17**, No. 3, 44 pp.
9. Fjørtoft, R., 1950: Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ.*, **17**, No. 6, 52 pp.
10. ——, 1951: The stability properties of large-scale atmospheric disturbances. (To appear in *Compendium of Meteorology*, American Meteorological Society.)
11. Fleagle, R. G., 1947: The fields of temperature, pressure, and three-dimensional motion in selected weather situations. *J. Meteor.*, **4**, 165–185.
12. Haurwitz, B., 1940: The motion of atmospheric disturbances on the spherical earth. *J. marine Res.*, **3**, 254–267.
13. Kuo, H. L., 1949: Dynamic instability of two-dimensional nondivergent flow in a barotropic atmosphere. *J. Meteor.*, **6**, 105–122.
14. Palmén, E., and C. W. Newton, 1951: On the three-dimensional motions in an outbreak of polar air. *J. Meteor.*, **8**, 25–39.
15. Rossby, C.-G., 1949: On a mechanism for the release of potential energy in the atmosphere. *J. Meteor.*, **6**, 163–180.
16. Rossby, C.-G., and collaborators, 1939: Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. marine Res.*, **2**, 38–55.
17. Smith, C. D., 1950: The destructive storm of November 25–27, 1950. *Mon. Wea. Rev.*, **78**, 204–209.
18. Southwell, R. V., 1946: *Relaxation methods in theoretical physics*. Oxford, Clarendon Press, 248 pp.
19. Sumner, E. J., 1950: The significance of vertical stability in synoptic development. *Quart. J. r. meteor. Soc.*, **76**, 384–391.
20. Sutcliffe, R. C., 1947: A contribution to the problem of development. *Quart. J. r. meteor. Soc.*, **73**, 370–383.