PHILOSOPHICAL TRANSACTIONS.

I. *Eddy Motion in the Atmosphere.*

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OUR knowledge of wind eddies in the atmosphere has so far been confined to the observations of meteorologists and aviators. The treatment of eddy motion in either incompressible or compressible fluids by means of mathematics has always been regarded as a problem of great difficulty, but this appears to be because attention has chiefly been directed to the behaviour of eddies considered as individuals rather than to the average effect of a collection of eddies. The difference between these two aspects of the question resembles the difference between the consideration of the action of molecule on molecule in the dynamical theory of gases, and the consideration of the average effect, on the properties of a gas, of the motion of its molecules.

It has been known for a long time that the retarding effect of the surface of the earth on the velocity of the wind must be due, in some way, to eddy motion; but apparently no one has investigated the question of whether any known type of eddy motion is capable of producing the distribution of wind velocity which has been observed by meteorologists, and no calculations have been made to find out how much eddy motion is necessary in order to account for this distribution. The present paper deals with the effect of a system of eddies on the velocity of the wind and on the temperature and humidity of the atmosphere. In a future paper the way in which they are produced and their stability when formed will be considered.

It is well known that wind velocity, temperature, and humidity vary much more rapidly in a vertical than in any horizontal direction, and further that the vertical component of wind velocity is very small compared with the horizontal velocity. It has been assumed, therefore, that the average condition of the air at any time is constant for a given height, over an area which is large compared with the maximum height considered. If u and v represent the components of undisturbed wind velocity parallel to horizontal axes, x and y, running from South to North and from West to East respectively, and if T and m represent the average temperature and the average amount of water vapour per cubic centimetre of air, this is equivalent to assuming that u, v, T, and m are functions of z, the height, and t, the time; and that they are independent of x and y.

Vertical Transference of Heat by Eddy Motion.

Let us first consider the propagation of heat in a vertical direction. The ordinary conductivity of heat by molecular agitation is so small that no sensible error will be introduced by leaving it out of the calculations. The only way in which large quantities of heat can be conveyed upwards or downwards through the atmosphere is by means of a vertical transference of air which retains its heat as it passes into regions where the temperature differs from that of the layer from which it started. If T′ and w' represent the temperature and the vertical component of the velocity of the air at any point, the rate at which heat is propagated across any horizontal area is $\iint \rho\sigma \mathrm{T}' w' \, dx\, dy$ where ρ and σ are the density and specific heat respectively, and the integral is taken over the area in question.

Since there is no vertical motion of the air as a whole $\iint \rho w' \, dx\, dy = 0$. Hence, in order that heat may be conveyed downwards, the air at any level must be hotter in a downward than in an upward current. In order that this may be the case the potential temperature* of the air must increase upwards. The excess of temperature in a downward current over the mean temperature at any level will depend partly on the vertical distance through which the air has travelled since it was at the same temperature as its surroundings, and partly on the rate of change in potential temperature with height. If the hot air, after crossing the horizontal area, continues on its downward course with undiminished velocity and without losing heat, and if the mean potential temperature continues to decrease downwards, the rate of transmission of heat across a horizontal area will continually increase. On the other hand, if the air returns across the area without losing its heat, there will be no resultant transmission of heat at all. The air must lose its heat by mixture with surrounding air after crossing the area.

Consider now the transference of heat across a large horizontal area A at a height z. Suppose that at a time t_0 an eddy broke away from the surrounding air at height z_0 and arrived at the point xyz at time t; z_0 and t_0 are then functions of x, y, z, and t. Suppose that initially the eddy had the same temperature as its surroundings. Let $\theta(z, t)$ be the average potential temperature of the air in the layer at height z at time t; then since the air preserves the potential temperature

* Potential temperature is the temperature air would assume if its volume were changed adiabatically till it was at some standard pressure, say 760 mm.

of the layer from which it originated, the potential temperature at the point xyz at time t is $\theta(z_0, t_0)$.

The amount of heat which passes per second across the area A is therefore $\rho\sigma \iint_A w\theta(z_0 t_0)\, dx\, dy$. Now $\theta(z_0, t_0)$ may be expressed in the form

$$\theta(z_0, t_0) = \theta(z, t) + (z_0 - z)\frac{\partial\theta}{\partial z} + (t_0 - t)\frac{\partial\theta}{\partial t},$$

provided that the changes in θ in the height $z - z_0$ and in the time $t - t_0$ are small compared with θ. Hence

$$\iint_A w\theta(z_0, t_0)\, dx\, dy = \theta(z, t)\iint_A w\, dx\, dy + \frac{\partial\theta}{\partial z}\iint_A w(z_0 - z)\, dx\, dy + \frac{\partial\theta}{\partial t}\iint_A w(t_0 - t)\, dx\, dy.$$

Now $t_0 - t$ is necessarily negative, and since $\iint_A w\, dx\, dy = 0$ it is evident that a positive value of w occurs as often as a negative one; hence if the eddy motion is uniformly distributed $\iint_A w(t_0 - t)\, dx\, dy = 0$.

The rate at which heat crosses the plane is therefore

$$-\rho\sigma\frac{\partial\theta}{\partial z}\iint_A w(z - z_0)\, dx\, dy.$$

The rate at which heat crosses an area A at height $z + \delta z$ is

$$\left(-\frac{\partial\theta}{\partial z} - \frac{\partial\theta^2}{\partial^2 z}\delta z\right)\iint_A w(z - z_0)\, dx\, dy \quad \text{for} \iint_A w(z - z_0)\, dx\, dy$$

does not vary with z if the eddy motion is uniformly distributed. Hence the rate at which heat enters the volume $A\delta z$ is

$$\rho\sigma\frac{\partial^2\theta}{\partial z^2}\delta z\iint_A w(z - z_0)\, dx\, dy.$$

Now since mixtures which take place within this volume merely alter the distribution of the heat contained in it without affecting its amount, this must be equal to $\rho\sigma\frac{\partial\theta}{\partial t}A\delta z$. Hence we obtain the equation for the propagation of heat by means of eddies in the form $\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2}\frac{1}{A}\iint_A w(z - z_0)\, dx\, dy$. But $\frac{1}{A}\iint_A w(z - z_0)\, dx\, dy$ is the average value of $w(z - z_0)$ over a horizontal area, hence it may be expressed in the form $\frac{1}{2}(\overline{w}d)$, where d is the average height through which an eddy moves from the layer at which it was at the same temperature as its surroundings, to the layer with which it mixes. $\overline{w}$ is defined by the relation $\frac{1}{2}(\overline{w}d) =$ average value of $w(z - z_0)$ over a horizontal plane; it roughly represents the average vertical velocity of the air in places where it is moving upwards. The divisor 2 is inserted because the air at any given point is equally likely to be in any portion of the path of an eddy, so that the average value of $z-z_0$ should be approximately equal to $\frac{1}{2}(d)$.

The equation for the propagation of heat by means of eddies may now be written

$$\frac{\partial\theta}{\partial t} = \frac{\overline{w}d}{2}\frac{\partial^2\theta}{\partial z^2} \quad \quad (1)$$

The equation for the propagation of heat in a solid of coefficient of conductivity κ, specific heat σ and density ρ is

$$\frac{\delta \mathrm{T}}{\delta t} = \frac{\kappa}{\rho\sigma}\frac{\delta^2 \mathrm{T}}{\delta z^2}.$$

It appears therefore that potential temperature is transmitted upwards through the atmosphere by means of eddies in the same way that temperature is transmitted in a solid of conductivity κ, provided $\kappa/\rho\sigma = \frac{1}{2}\overline{w}d$. We shall in future call κ the "eddy conductivity" of the atmosphere.

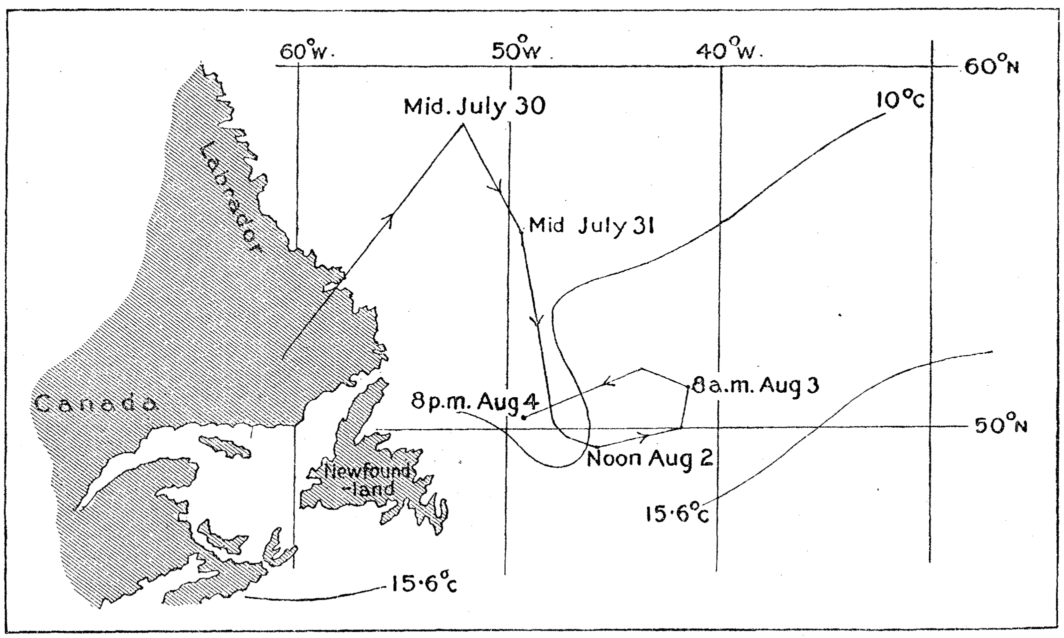
If we know the temperature distribution at any time (say $t = 0$), and if we know the subsequent changes of temperature at the base of the atmosphere we can calculate, on the assumption of a uniform value for $\kappa/\rho\sigma$, the temperature distribution at any subsequent time. Conversely, if the temperature distribution on two occasions be known, and if we know the temperature of the base of the atmosphere at all intermediate times, we can obtain some information about the coefficient of eddy conductivity, and hence about the eddies themselves.

I was fortunate enough to be able to obtain the necessary data on board the ice-scout ship "Scotia" in the North Atlantic last year. On several occasions the distribution of temperature in height was determined by means of kites. The temperature changes experienced by the lowest layer of the air as it moved up to the position where its temperature distribution was explored by means of a kite, were found in the following way. The path of the air explored in the kite ascent was traced back through successive steps on a chart by means of observations of wind velocity and direction taken on board English, German, and Danish vessels, which happened to be near the position occupied by the air at various times previous to the kite ascent. This method was adopted by Shaw and Lempfert in their work on the 'Life History of Surface Air Currents.' It depends for success on being able to obtain observations in the right spot at the right time. It frequently happens that no such observations are obtainable, and in these cases it is impossible to proceed with the investigation. Owing to this difficulty I was unable to trace the air paths for more than seven of the ascents.

Having obtained the path of the air, the next step is to find the temperature of the sea below it. This is a comparatively easy matter, for a careful watch is kept by the liners on the temperature of the North Atlantic. The results of their observations are plotted by the Meteorological Office on weekly charts, on which isothermal lines are drawn to represent sea temperatures of 80° F., 70° F., 60° F., 50° F., and 40° F. These charts are published on a small scale in the weekly weather report of the Meteorological Office, but Captain Campbell Hepworth was kind enough to lend me the originals, and on them I plotted the air paths.

One of the charts, with the air's path marked on it, is shown in fig. 1.*

It has been found that the temperature of the air rarely differs from that of the surface of the sea by more than 2° C., and usually the difference is only a fraction of



Path of air and sea temperature for kite ascent of August 4th.

Fig. 1.

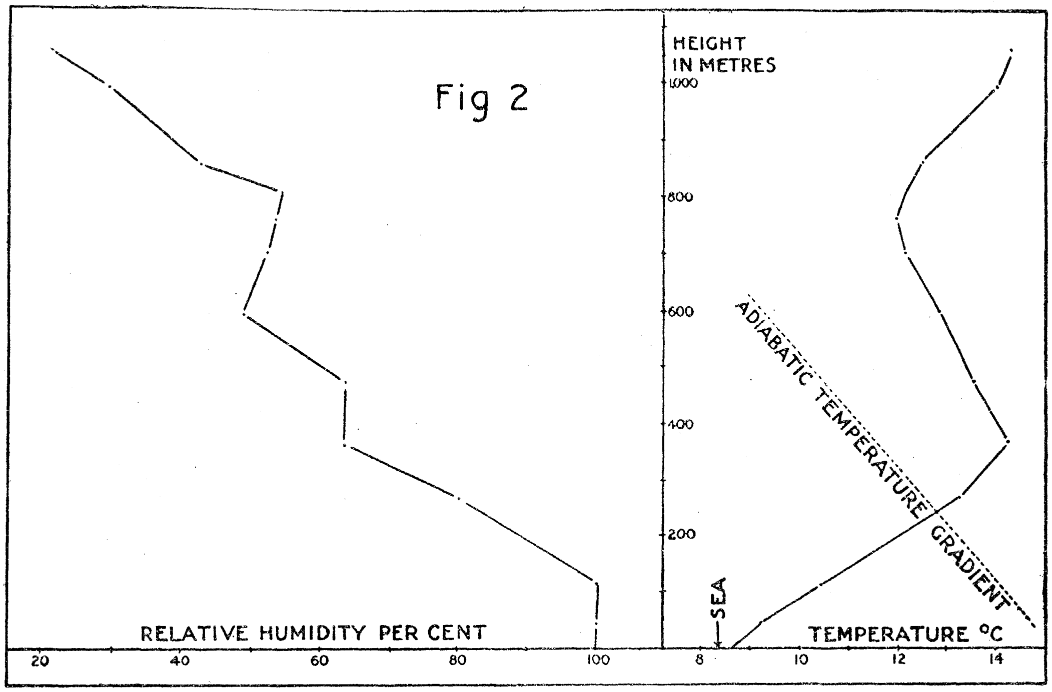
a degree. The temperature of the base of the atmosphere at any point along the air's path has, therefore, been assumed to be that of the surface of the sea. In many of the kite ascents the temperature of the sea, and therefore of the surface air, increased up to a certain point along the air's path and then began to decrease. While the air was moving along the first part of the path its temperature might be expected to decrease with height at a rate greater than the adiabatic rate.† When

* Others are reproduced in the 'Report of the "Scotia" Expedition, 1913.'

† If the temperature of the air diminishes at the adiabatic rate of 10° C. per kilometre, its potential temperature is constant, so that no amount of eddy motion can transfer heat either upwards or downwards.

the air entered the portion of the path along which temperature was diminishing it might be expected that the cooling effect of the sea would not spread upwards instantaneously, but that it would make its way gradually into the upper layers. We might expect, therefore, that, if a kite were to be sent up into the air as it was passing over the second part of its path, the temperature would increase up to a certain height; and that, above that height, it would have the temperature gradient which it had acquired during its passage along the first portion of its path.

If a curve be drawn to represent the temperature of the atmosphere at different heights a change from heating to cooling along the air's path will give rise to a corresponding bend in this curve. The height of this bend above the surface of the earth will depend partly on the interval which elapsed between the time when the air was passing over the portion of the path where heating stopped and cooling began



and the time of the ascent, and partly on the eddy conductivity of the atmosphere. If we know two of these quantities we should be able to calculate the third.

On the right hand side of fig. 2 is shown the temperature distribution at various heights from the surface up to 1100 metres in the case of the air which had blown along the path drawn on the chart shown in fig. 1. It will be seen that there are two bends in the curve. The lowest portion from the surface up to 370 metres evidently corresponds with the cooling of the lowest strata of the atmosphere which had been going on ever since the air turned back from the warm water of the Atlantic towards the cold water of the Great Bank of Newfoundland.

The air explored in the ascent of August 4th turned towards the west at 8 a.m. on August 3rd and continued blowing on to colder and colder water till the time of the ascent, 8 p.m., August 4th. It appears, therefore, that the cooling had extended upwards through a height of 370 metres in 36 hours. An arrow has been drawn on the base line to represent the temperature of the sea which, as we should expect, is slightly less than the temperature of the air which is being cooled by it. The portion of the temperature curve of fig. 2 which lies between 370 metres and 770 metres is due to the warming which the air had undergone between the evening of July 30th and 8 a.m., August 3rd. The portion of the curve above 770 metres to which the warming of July 30th to August 3rd had not yet reached, is due to the cooling which the air experienced as it blew off the warm land of Canada on to the cold Arctic water which runs down the coast of Labrador.

The curve on the left hand of fig. 2 represents the humidity of the atmosphere at different heights. It is reproduced here for two reasons, firstly, the extreme dryness of the air at 1100 metres (the humidity being only 20 per cent.) shows that the air really had blown off the land as is shown on the chart in fig. 1; and, secondly, because it shows that changes in the amount of water vapour in the atmosphere are propagated upwards in the same way as changes in temperature. Bends in the humidity curve occur at the same heights as bends in the temperature curve. This is in fact to be expected, for it is evident that the reasoning which was used to deduce the equation (1) would serve equally well to deduce an equation $\frac{\partial m}{\partial t} = \frac{\overline{w}d}{2}\frac{\partial^2 m}{\partial z^2}$ for the propagation of water vapour into the atmosphere.

Temperature-height curves, similar to that shown in fig. 2, were traced for all the kite ascents which were made from the "Scotia," and most of them did have bends in them. In all cases in which it was possible to trace the air's path a bend in the curve was found to correspond, either to a change from heating to cooling (or *vice versâ*) of the surface air as it moved along its path, or to a sudden change in the rate of cooling when the air crossed the sharply defined edge of the Gulf Stream.

In most cases the change from heating to cooling was due to a change in the direction of the wind. Changes in wind direction occur simultaneously over large areas of the ocean, hence, even if the exact position of the path is not accurately determined, we may be able to obtain reliable information as to the time at which heating ceased and cooling began; and calculations which depend on the interval between the time of this change and the time of the kite ascent will be more accurate than those which involve the length or position of the path.

Let us consider the temperature distribution in the atmosphere in an ideal case so chosen as to represent as nearly as possible the actual conditions of some of the "Scotia" kite ascents.

Suppose that the initial potential temperature of the atmosphere is taken to be zero at all heights, and suppose that the surface layers begin to be cooled at time $t = 0$ in such a way that the potential temperature θ_0 at the ground, $z = 0$, is a function $\phi(t)$ of the time, so that $\theta_0 = \phi(t)$.

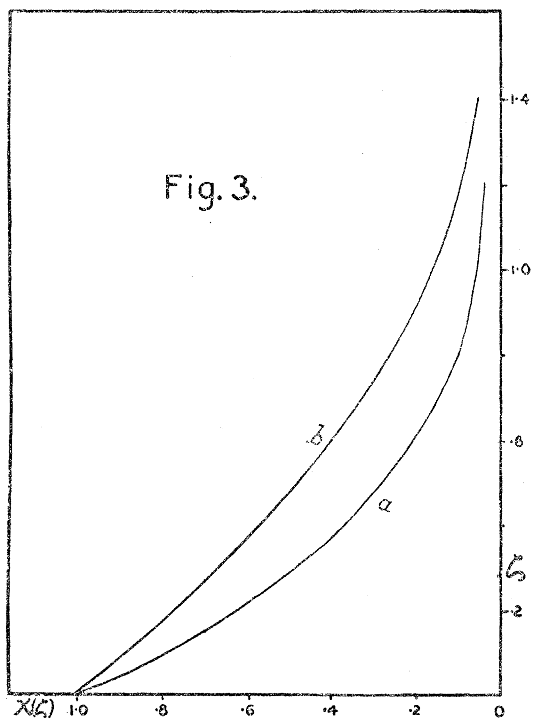
The solution of $\frac{\partial\theta}{\partial t} = \frac{\overline{w}d}{2}\frac{\partial^2\theta}{\partial z^2}$ which fits these conditions is*

$$\theta = \frac{2}{\sqrt{\pi}}\int_{z(2\overline{w}dt)^{-\frac{1}{2}}}^{\infty}\phi\left(t-\frac{z^2}{2\overline{w}d\mu^2}\right)e^{-\mu^2}\,d\mu.$$

The two following cases are of interest :—

(*a*) The surface temperature decreases uniformly as t increases at a rate of p° C. per second, so that $\theta_0 = -pt$.

(*b*) The temperature of the surface layers changes suddenly from 0 to θ_0 and afterwards remains constant.



In (*a*) the integral becomes

$$\theta = -pt\left[(1+2\zeta^2)\left(1-2\pi^{-\frac{1}{2}}\int_0^{\zeta}e^{-\mu^2}\,d\mu\right)-2\pi^{-\frac{1}{2}}\zeta e^{-\zeta^2}\right]$$
$$= \theta_0\psi(\zeta), \text{ say}$$

where $\zeta = z(2w'dt)^{-\frac{1}{2}}$ and $\psi(\zeta)$ represents the expression in square brackets.

The curve (*a*) in fig. 3 represents the values of ψ for values of ζ ranging from 0 to 1·2. It will be seen that when $\zeta = $ ·8 the value of ψ is $\frac{1}{10}$th of its value at the surface, where $\zeta = 0$.

* See 'FOURIER'S Series and Integrals,' H. S. CARSLAW, p. 238.

In (b) the integral becomes

$$\theta = \theta_0 \left(1 - 2\pi^{-\frac{1}{2}} \int_0^{\zeta} e^{-\mu^2}\, d\mu\right) = \theta_0 \chi(\zeta),$$

where $\chi(\zeta)$ represents the expression in brackets and ζ has the same meaning as before.

The curve (b) in fig. 3 represents the values of $\chi(\zeta)$ for different values of ζ. It will be seen that when $\zeta = 1\cdot2$ the temperature is about $\frac{1}{10}$th of the surface temperature θ_0. In actual cases it is not easy to say whether (a) or (b) is a better representation of the changes in temperature along the air's path. In most cases probably (a) is the best, but, in one case, that of the ascent of May 3rd, the bend in the temperature-height curve was due to the passage of the air across a sharply defined boundary between the warm waters of the Gulf Stream and the cold arctic water over the great Bank of Newfoundland,* and then one might expect (b) to be a truer representation of the vertical temperature distribution. In either case we shall not be far wrong if we assume that the height to which the new conditions have reached at time t is given by $\zeta = 1\cdot0$ or

$$z^2 = 2\overline{w}dt. \qquad (2)$$

If we can measure z, the height of the bend in the temperature-height curve, and if we know t, the interval which has elapsed between the time at which the rate of change of surface temperature along the air's path suddenly altered its value and the time of the ascent, the equation (2) enables us to calculate $\kappa/\rho\sigma$ or $\frac{1}{2}(\overline{w}d)$. The error in this result may be as great as 30 per cent., but it does at any rate give a good idea of the magnitude of the coefficient of eddy conductivity and of the amount of eddy motion which is necessary in order to produce the vertical temperature distributions which have been observed.

In some of the cases the potential temperature before the change which caused the bend in the temperature-height curve was not constant at all heights. In the case of the upper bend in the curve shown in fig. 2, for instance, the potential temperature increased with height before the warming which produced the upper band occurred. This, however, makes no difference to the rate at which the bend is propagated upwards. It is evident that if θ_1 and θ_2 be two solutions of $\dfrac{\partial\theta}{\partial t} = \dfrac{\overline{w}d}{2}\dfrac{\partial^2\theta}{\partial z^2}$, then $\theta_1 + \theta_2$ is also a solution. If the initial potential temperature before the change were $\theta = T_0 + \alpha z$, and if the surface temperature were to change suddenly to T_1 at time $t = 0$, the temperature at height z at a subsequent time t would be

$$\theta_0 = T_0 + \alpha z + T\ (T_1 - T_0)\,\chi(\zeta).$$

It is evident that the term $T_0 + \alpha z$ does not affect the rate at which the bend in the temperature-height curve is propagated upwards.

* See 'Reports of the "Scotia" Expedition, 1913.'

In Table I. the values of z and t observed in the "Scotia" kite ascents are given. In the first column is given the date of the ascent, in the second column the height of the bend in the temperature-height curve, in the third column the interval between the time of the change in the temperature conditions which give rise to the bend in the temperature-height curve and the time of the ascent, and in the fourth column are given values of $\frac{1}{2}(\overline{w}d)$ in C.G.S. units, calculated from the equation $\frac{1}{2}(\overline{w}d) = z^2/4t$.

TABLE I.

Date of observation.	z, metres.	t, hours.	$\frac{z^2}{4t}$ in C.G.S. units.	Average wind force (Beaufort scale).
May 3rd	270	15	$3 \cdot 4 \times 10^3$	3·3
July 17th	140	24	$\cdot 57 \times 10^3$	2
July 25th	610	168	$1 \cdot 5 \times 10^3$	2 to 3
July 29th	170	15	$1 \cdot 3 \times 10^3$	2·2
August 2nd	200	11	$2 \cdot 5 \times 10^3$	3
August 4th (two bends in the temperature-height curve)	370	36	$2 \cdot 6 \times 10^3$	2·5
	770	120	$3 \cdot 4 \times 10^3$	3·1

It will be seen that the values of $\frac{1}{2}(\overline{w}d)$ vary through a large range. It is to be expected that the amount of eddy motion will depend on the wind velocity; accordingly, a fifth column has been added to show the average wind force during the time t. The figures in this column are the means of the Beaufort wind-force numbers recorded at the "Scotia" during a time t before the ascent. In the case of the ascent of July 25th the necessary observations were unobtainable because the "Scotia" was in port till July 24th. In this case the wind force recorded by the steamers, from whose observations was traced the path along which the air approached the position of the "Scotia" at the time of the kite ascent of July 25th, varied from 2 to 3 on the Beaufort Scale.

It will be seen that for the ascents of May 3rd, August 2nd, and August 4th, when the wind force was about 3, the values of $\frac{1}{2}(\overline{w}d)$ are $3 \cdot 4 \times 10^3$, $2 \cdot 5 \times 10^3$, $2 \cdot 6 \times 10^3$

and $3{\cdot}4 \times 10^3$; and that for July 17th and July 29th, when the wind force was about 2, the values were very much lower, being $0{\cdot}57 \times 10^3$ and $1{\cdot}3 \times 10^3$ respectively.

The fact that these figures are so consistent, although t varies from 11 hours to 7 days and z from 140 metres to 770 metres, seems to indicate that the eddy motion does not diminish to any great extent in the first 770 metres above the surface.

Vertical Change of Velocity due to Eddy Motion.

In the first part of this paper the vertical transference of heat by means of eddies has been discussed. For this purpose it was necessary to consider only the vertical component of eddy velocity, but in the questions which are treated in the succeeding pages it is no longer possible to leave the horizontal components out of the calculations. It seems natural to suppose that eddies will transfer not only the heat and water vapour, but also the momentum of the layer in which they originated to the layer with which they mix. In this way there will be an interchange of momentum between the different layers. If U_z and V_z represent the average horizontal components of wind velocity at height z parallel to perpendicular co-ordinates x and y, and if u', v', w' represent components of eddy velocity so that the three components of velocity are U_z+u', V_z+v' and w', then the rate at which x-momentum is transmitted across any horizontal area is

$$\iint \rho \left(U_z+u'\right) w' dx\, dy, \qquad \qquad (3)$$

and the rate at which y-momentum is transferred is $\iint \rho \left(V_z+v'\right) w' dx\, dy$ the integrals extending over the area in question.

If we were to suppose that an eddy conserves the momentum of the layer in which it originated so that $U_z+u' = U_{z_0}$ and $V_z+v' = V_{z_0}$, where z_0 is the height of the layer in question, we could obtain the values of the integrals in the same way that we did in the case of heat transference. In the case of heat transference, owing to the small value of the ordinary coefficient of "molecular" conductivity, the only way in which an eddy can lose its temperature is by mixture; but in the case of transference of momentum the eddy can lose or gain velocity owing to the existence of local variations in pressure over a horizontal plane. Such variations are known to exist; they are in fact a necessary factor in the production of disturbed motion, and they enter into all calculations respecting wave motion. We cannot, therefore, leave them out of our calculations without further consideration, though it will be seen that they probably do not affect the value of the integral (3) when it is taken over a large area.

Consider a particular case of disturbed motion. Suppose that the fluid is incompressible and that the motion takes place in two dimensions x and z. Suppose that originally the fluid is flowing parallel to the axis of x with velocity U_z and that the disturbance has arisen from dynamical instability, or from disturbances transmitted from the surface of the earth. The rate at which x-momentum leaves a layer of thickness δz is

$$\left[\frac{\partial}{\partial z}\iint \rho\,(\mathrm{U}_z+u')\,w'\,dx\,dy\right]\delta z.$$

But U_z is constant over the plane xy and since there is no resultant flow of fluid across a horizontal plane $\iint \rho \mathrm{U}_z w' dx\,dy = \mathrm{U}_z \iint \rho w' dx\,dy = 0$.

Hence, if we write I for the value of the expression in square brackets

$$\mathrm{I} = \frac{\partial}{\partial z}\iint \rho\,(\mathrm{U}_z+u')\,w' dx\,dy = \rho\frac{\partial}{\partial z}\iint u'w' dx\,dy$$

$$= \rho\iint\left(u'\frac{\partial w'}{\partial z}+w'\frac{\partial u'}{\partial z}\right)dx\,dy. \qquad (4)$$

The equation of continuity is

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0. \qquad (5)$$

Since the motion is confined to two dimensions

$$\frac{\partial\,(\mathrm{U}_z+u')}{\partial z}-\frac{\partial w'}{\partial x} = \text{twice the vorticity of the fluid at the point } (x, y, z).$$

And since every portion of the fluid retains its vorticity throughout the motion, this must be equal to twice the vorticity which the fluid at the point (x, y, z) had before the disturbance set in. This is equal to the value of $(d\mathrm{U}_z/dz)$ at the height, z_0,* of the layer from which the fluid at the point (x, y, z) originated. If this value be expressed by the symbol $[d\mathrm{U}_z/dz]_{z_0}$ we see that the dynamical equations of fluid motion lead to the equation

$$\frac{d\mathrm{U}_z}{dz}+\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}=\left[\frac{d\mathrm{U}_z}{dz}\right]_{z_0}. \qquad (6)$$

Substituting in (4) the values of $\frac{\partial w'}{\partial z}$ and $\frac{\partial u'}{\partial z}$ given by (5) and (6), we find

$$\mathrm{I} = \rho\iint\left(-u'\frac{\partial u'}{\partial x}+w'\frac{\partial w'}{\partial x}+w'\left[\frac{d\mathrm{U}_z}{dz}\right]_{z_0}-w'\frac{d\mathrm{U}_z}{dz}\right)dx\,dy$$

$$= \rho\iint \rho\frac{\partial\,(w'^2-u'^2)}{\partial x}dx\,dy+\rho\iint w'\left(\left[\frac{d\mathrm{U}_z}{dz}\right]_{z_0}-\frac{d\mathrm{U}_z}{dz}\right)dx\,dy. \qquad (7)$$

The first term integrates and vanishes when a large area is considered; but the second term does not vanish.

* z_0 is evidently a function of x and z when the motion is confined to two dimensions.

To find the value of the second term expand $[dU_z/dz]_{z_0}$ in powers of $z_0 - z$.

$$\left[\frac{dU_z}{dz}\right]_{z_0} = \frac{dU_z}{dz} + (z_0 - z)\frac{d^2U_z}{dz^2} + \frac{(z_0-z)^2}{2}\frac{d^3U_z}{dz^3} + \ldots .$$

Hence (7) becomes

$$\mathrm{I} = \rho \iint w' \left[(z_0 - z)\frac{d^2U_z}{dz^2} + \frac{(z_0-z)^2}{2}\frac{d^3U_z}{dz^3} + \ldots\right] dx\, dy. \quad . \quad . \quad . \quad (8)$$

So far nothing has been said about the magnitude of the disturbance; (8) is true even if the disturbance be large. Let us now suppose that the height, $z - z_0$, through which any portion of the fluid has moved from its undisturbed position is of such a magnitude that the change in dU_z/dz in that height is small compared with dU_z/dz itself. In that case (8) becomes

$$\mathrm{I} = \rho \frac{d^2U_z}{dz_2} \iint w' (z_0 - z)\, dx\, dy.$$

The rate at which x-momentum leaves a layer of thickness δz is therefore

$$\rho \frac{\partial^2 U_z}{\partial z^2} \delta z \iint w' (z - z_0)\, dx\, dy.$$

The effect of the disturbance is therefore to reduce the x-momentum of a horizontal layer of thickness δz at rate $\rho \frac{\partial^2 U_z}{\partial z^2} \delta z \times$ [average value of $w' (z_0 - z)$] per unit area.

The same effect would be produced on a layer of thickness δz by a viscosity equal to $\rho \times$ average value of $w' (z - z_0)$ if the motion had not been disturbed. If, some time after the disturbance has set in, all the air at any level mixes, no change will take place in the average momentum of the layer. Deviations from the mean velocity of the layer will disappear, and the velocity will be horizontal once more and uniform over any horizontal layer. When, therefore, we wish to consider the disturbed motion of layers of air, we can take account of the eddies by introducing a coefficient of eddy viscosity equal to $\rho \times$ average value of $w' (z - z_0)$, and supposing that the motion is steady, $z - z_0$ is the height through which air has moved since the last mixture took place.

As before in the case of the eddy conduction of heat, we can express the average value of $w' (z - z_0)$ in the form $\frac{1}{2}(\overline{w}d)$, where d is the average height through which an eddy moves before mixing with its surroundings, and $\overline{w}$ roughly represents the average vertical velocity in places where w' is positive. It will be noticed that the value we have obtained for eddy-viscosity is the same as that which we would have obtained if we had neglected variations in pressure over a horizontal plane, and had assumed that air in disturbed motion conserves the momentum of the layer from which it originated till it mixes with its new surroundings, just as it conserves its potential temperature. Whether this result is true when the disturbance takes place in three dimensions, I have been unable to discover.

If it is true, there is a relation $\kappa/\rho\sigma = \mu/\rho = \frac{1}{2}(\overline{w}d)$ between κ the eddy conductivity and μ the eddy viscosity; if any method of deducing μ from meteorological observations could be found, it would be possible to verify the relation numerically.

Relation of Observed Velocity to Gradient Velocity.

We may expect to discern the effect of eddy viscosity in cases where the wind velocity changes with altitude, and where the force due to eddy viscosity prevents the wind from attaining the velocity which we should expect on account of the pressure distribution. These conditions arise near the surface of the earth. The velocity and direction which we should expect on account of the pressure distribution, are called the gradient velocity and the gradient direction. In general, the wind near the ground falls short of the gradient velocity by about 40 per cent., and the direction near the ground is about 20 degrees from the gradient direction. At a height which varies on land from 200 to 1000 metres the wind becomes equal both in velocity and in direction to the gradient wind.

Let us consider the motion of air over the earth's surface under the action of a constant pressure gradient G acting in the direction of the axis of y. The equations of motions of an imcompressible* viscous fluid are†

$$\left.\begin{aligned} \frac{\mathrm{D}u}{\mathrm{D}t} &= \mathrm{X} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u \\ \frac{\mathrm{D}v}{\mathrm{D}t} &= \mathrm{Y} - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\nabla^2 v \\ \frac{\mathrm{D}w}{\mathrm{D}t} &= \mathrm{Z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 w \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (9)$$

where u, v, w are components of velocity parallel to the co-ordinates x, y, z; p is the pressure, and X, Y, Z, are the components of the external forces on unit mass of the fluid.

The forces acting are the force due to the earth's rotation and gravity.

Hence

$$\left.\begin{aligned} \mathrm{X} &= -2\omega v \sin\lambda \\ \mathrm{Y} &= 2\omega u \sin\lambda \\ \mathrm{Z} &= -g \end{aligned}\right\} \text{where } \omega \text{ is the angular velocity of the earth's rotation and } \lambda \text{ is the latitude.}$$

The pressure is given by $p = \text{constant} - g\rho z + \mathrm{G}y$.

* The atmosphere is not incompressible, but compressibility makes no difference in the present instance.

† See Lamb's 'Hydrodynamics,' p. 338.

If we assume the motion to be horizontal equations (9) become

$$0 = -2\omega v \sin\lambda + \frac{\mu}{\rho}\frac{d^2u}{dz^2}, \quad \ldots \quad (10)$$

$$0 = -2\omega u \sin\lambda - \frac{G}{\rho} + \frac{\mu}{\rho}\frac{d^2v}{dz^2}. \quad \ldots \quad (11)$$

Eliminating u the equation for v becomes

$$\frac{d^4v}{dz^4} + 4B^4 v = 0 \qquad \text{where } 2B^2 = \frac{2\omega\rho\sin\lambda}{\mu}.$$

Taking into consideration the fact that v does not become infinite for infinite values of z, the solution of this is

$$v = A_2 e^{-Bz}\sin Bz + A_4 e^{-Bz}\cos Bz. \quad \ldots \quad (12)$$

Differentiating this twice with respect to z we find

$$\frac{d^2v}{dz^2} = 2B^2(-A_2 e^{-Bz}\cos Bz + A_4 e^{-Bz}\sin Bz).$$

Substituting this value in (11) we find

$$u = A_2 e^{-Bz}\cos Bz - A_4 e^{-Bz}\sin Bz + \frac{G}{2\mu B^2}. \quad \ldots \quad (13)$$

The quantity $\frac{G}{2\mu B^2}$ or $\frac{G}{2\omega\rho\sin\lambda}$ is the gradient velocity, so that at great heights, $v = 0$ and u is equal to the gradient velocity.

The values of A_2 and A_4 will be found by imposing suitable boundary conditions. If there is slipping at the earth's surface it seems natural to assume that it is in the direction of the stress in the fluid. In this case one boundary condition will be

$$\left[\frac{du/dz}{u}\right]_{z=0} = \left[\frac{dv/dz}{v}\right]_{z=0}. \quad \ldots \quad (14)$$

Where the square brackets are intended to show that the values of the quantities contained in them are to be taken at the surface of the ground, $z = 0$.

Substituting for u, v, du/dz and dv/dz, and putting $z = 0$, equation (14) becomes

$$-\left(\frac{A_2 + A_4}{A_2 - A_4}\right) = \frac{A_2 + \frac{G}{2\mu B^2}}{A_4} = \frac{A_2 + Q_G}{A_4}, \quad \ldots \quad (15)$$

where Q_G represents the gradient velocity.

In order to determine the motion completely one more relation between A_2 and A_4 is necessary. Let the wind at the surface be deviated through an angle α from the gradient wind in such a way that if one stands facing the surface wind the gradient wind will be coming from the right if α be positive.

Then

$$\tan\alpha = -\left[\frac{v}{u}\right]_{z=0} = \frac{A_4}{A_2+Q_G} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Solving (15) and (16) for A_2 and A_4

$$A_2 = \frac{-\tan\alpha\,(1+\tan\alpha)}{1+\tan^2\alpha}\,Q_G,$$

$$A_4 = \frac{-\tan\alpha\,(1-\tan\alpha)}{1+\tan^2\alpha}\,Q_G.$$

The surface wind, which we may denote by Q_S is equal to

$$\sqrt{[u^2+v^2]_{z=0}} = \sqrt{A_4^2+(A_2+Q_G)^2}$$

$$= \frac{Q_G}{1+\tan^2\alpha}\sqrt{\tan^2\alpha\,(1-\tan\alpha)^2+(1-\tan\alpha)^2},$$

or

$$Q_S = Q_G(\cos\alpha - \sin\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It is interesting to compare the value given by (17) for the ratio of Q_S to Q_G with the value, $\cos\alpha$, given by GULDBERG and MOHN* for the same ratio, and with the most recent observations of wind velocity at different altitudes above the surface of the earth.

Mr. G. M. B. DOBSON of the Central Flying School at Upavon has recently published† the results of a number of observations made by means of pilot balloons over Salisbury Plain, which is an excellent place for such observations on account of its open situation. He finds that α is smaller for light winds than for strong winds, and he accordingly divides up his ascents into three classes, those which took place in light winds, when the velocity of the wind at a height of 650 metres is below 4·5 metres per second, those in moderate winds between 4·5 and 13 metres per second, and those in strong winds above 13 metres per second.

The comparison is shown in Table II. It will be seen that the observed deviation of the surface wind from the gradient direction agrees well with the theory we have been considering, but not with the theory of GULDBERG and MOHN.

The agreement between theory and observation is, however, more striking in another respect. The deviation of the direction of the wind at any height from the

* 'Studies of the Movements of the Atmosphere,' 1883–85. An English translation appears in "The Mechanics of the Earth's Atmosphere," by CLEVELAND ABBE; 'Smithsonian Miscellaneous Collections,' 1910.

† 'Quarterly Journal of the Royal Meteorological Society,' April, 1914.

gradient direction is due to the retarding of the wind velocity below the gradient velocity by friction or by viscosity. One might expect, therefore, that the wind would attain the gradient direction at the same height as the gradient velocity. This would, in fact, follow from the theory of GULDBERG and MOHN. Most observations have failed to give reliable information on this point, partly because irregularities on the surface of the earth have introduced complicated conditions, which cannot be taken account of, and partly because the observations have not been grouped according to the wind velocity.* Neither of these objections applies to Mr. DOBSON's observations. Salisbury Plain, though inferior to the sea, is as good a place for wind

TABLE II.

	Light winds.	Moderate winds.	Strong winds.
Observed value of $\frac{Q_S}{Q_G}$.	·72	·65	·61
Observed angle α.	13 degrees	$21\frac{1}{2}$ degrees	20 degrees
α, calculated from (17) so as to correspond with the observed value of $\frac{Q_S}{Q_G}$.	14 degrees	18 degrees	20 degrees
α, calculated from GULDBERG and MOHN's theory so as to correspond with the observed value of $\frac{Q_S}{Q_G}$.	44 degrees	49 degrees	52 degrees

observations near the surface as one could find on land; and as has been explained already, his results are grouped according to wind velocity. Mr. DOBSON finds that the gradient direction is not attained till a height is reached which is more than twice the height at which the gradient velocity is first attained. He remarks, in fact, that the gradient velocity is usually attained at a height of 300 metres, though the gradient direction is not found till a height of 800 metres has been attained. This is a most remarkable result, but it might have been expected from the equations (12) and (13). The height at which the gradient direction is attained is given by putting $v = 0$ in (12). If H_1 be the height in question

$$A_2 \sin BH_1 + A_4 \cos BH_1 = 0$$

* Owing to the fact that μ/ρ depends on the wind force we should evidently expect more consistent results when the observations are grouped according to wind velocity.

so that

$$\tan BH_1 = -\frac{A_4}{A_2}$$

Substituting for A_2 and A_4 their values in terms of α

$$\tan BH_1 = -\frac{1-\tan\alpha}{1+\tan\alpha} = \tan\left(\alpha-\frac{\pi}{4}\right). \quad (18)$$

Since α is positive and less than $\frac{\pi}{4}$ the smallest positive value of H_1 is given by

$$BH_1 = \frac{3\pi}{4}+\alpha. \quad (19)$$

The height H_2 at which the wind velocity first becomes equal to the gradient velocity is given by $u^2+v^2 = Q_G{}^2$. This reduces to

$$e^{-BH_2} = \frac{(1+\tan\alpha)\cos BH_2-(1-\tan\alpha)\sin BH_2}{\tan\alpha} \quad (20)$$

Equation (20) can be solved so as to give $\tan\alpha$ in terms of BH_2, and when several corresponding values of α and BH_2 have been obtained BH_2 can be obtained by interpolation in terms of α. In Table III. are shown the values of BH_1 and BH_2 and H_1/H_2 corresponding to values of α from 0 to 45 degrees.

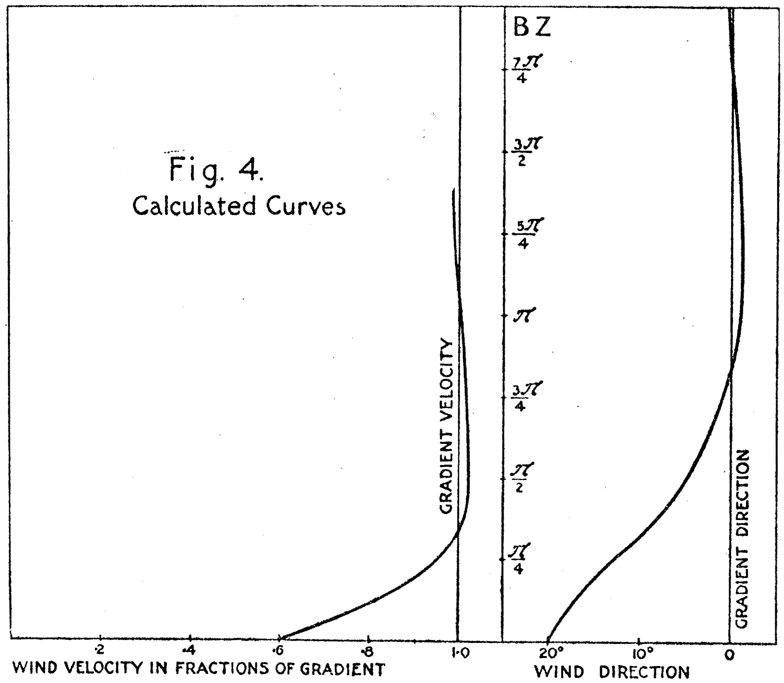
TABLE III.

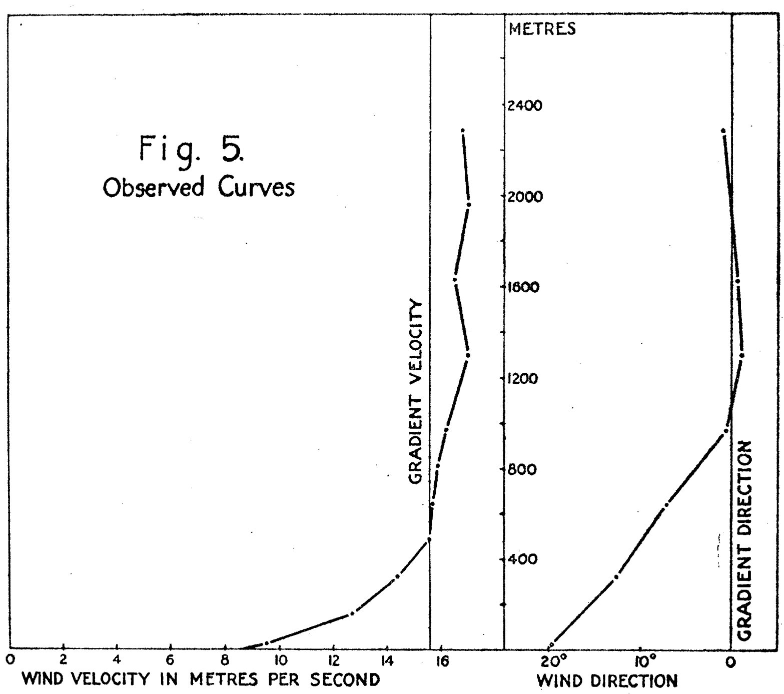
α.	BH_2.	BH_1.	$\frac{H_1}{H_2}$.
0 degrees	·78	2·35	3·0
10 degrees	·91	2·53	2·8
20 degrees	1·04	2·70	2·6
30 degrees	1·20	2·88	2·4
45 degrees	1·44	3·15	2·2

It appears, therefore, that H_1/H_2 varies from 3 to 2·2.

Mr. DOBSON gives $\frac{800 \text{ metres}}{300 \text{ metres}} = 2·66$ as the observed value of H_1/H_2, and his values of α all were about 20 degrees. It is probably a coincidence that the observed ratio, 2·66, should be so very close to the calculated ratio 2·6, but the coincidence is at least significant.

In order more easily to compare the theoretical results with the observations the curves shown in fig 4. have been prepared. Fig. 4 shows the way in which wind





velocity and direction vary with height in the theoretical case we have been considering when $\alpha = 20$ degrees. Fig. 5 is reproduced by permission of Mr. Dobson. It represents the observed velocity and direction of strong winds at different heights. In each of the figures the curve on the right represents deviations from the gradient direction, which is shown as a vertical line. The curve on the left represents wind velocity at different heights.

It will be seen that there is good agreement between the two sets of curves. Strong winds have been chosen for the comparison in preference to light winds, because it is less likely that heat-convection currents will persist through such a distance before mixing takes place, as to prevent the resistance, due to eddy motion, from obeying the ordinary laws of viscosity. The observed curves for light winds, however, agree as well with the theoretical curves as those for strong winds.

Besides the various points of resemblance already noticed between theory and observation, an inspection of the curves in figs. 4 and 5 reveals yet another. Above the height at which the gradient direction is attained the wind goes on veering slightly up to a certain height, when it begins to return again to the gradient direction. The wind is again blowing along the gradient direction at a height slightly less than twice the height at which it first attained it. Nearly all the curves in Mr. DOBSON's paper have this characteristic sinuosity, but they are not the only ones which show it. Mr. J. S. DINES, in his Third Report to the Advisory Committee for Aeronautics (1912), has published a number of curves which exhibit the same sort of sinuosity. The theoretical curve, fig 4, has the same characteristic. The successive heights H_1, H'_1, H''_1, ... at which the wind blows exactly along the gradient direction are given by the solutions of equation (18).

We have already obtained the first solution, namely $BH_1 = \dfrac{3\pi}{4}+\alpha$.

The others are $BH'_1 = \dfrac{3\pi}{4}+\alpha+\pi$, $BH''_1 = \dfrac{3\pi}{4}+\alpha+2\pi$ and ... .

The ratio of the first two is $\dfrac{H'_1}{H_1} = \dfrac{\frac{1}{4}(7\pi)+\alpha}{\frac{1}{4}(3\pi)+\alpha}$.

When $\alpha = 20$ degrees this is equal to 2·16.

In the case of strong winds it will be seen from fig. 5 that the observed values of H_1 and H'_1 are 900 metres and 1750 metres. Hence the observed value of H'_1/H_1 is 1·95. The good agreement between the observed and calculated values of H'_1/H is possibly a coincidence, but it is interesting to notice that, on theoretical grounds, we should expect a sinuosity in the curve representing the direction of the wind at various heights when it blows under the action of a constant pressure gradient, and that such a sinuosity is actually observed.

The close agreement between theory and observation is evidence that the assumptions made in the theory are correct. In particular the eddy motion does not diminish much in the first 900 metres.

We have seen that

$$BH_1 = \frac{3\pi}{4} + \alpha.$$

If, therefore, we can measure α and H_1 we can calculate B. The commonest value for α on land is 20 degrees, in fact, for all except light winds, it is near to 20 degrees. In the kite ascents on the "Scotia" the wind usually veered two points ($22\frac{1}{2}$ degrees) in the first 100 or 200 metres and after that remained constant in direction at greater heights. It appears, therefore, that on the sea also α is about 20 degrees. Assuming then that $\alpha = 20$ degrees, we see from Table III. that $BH_1 = 2{\cdot}7$.

Substituting for B its value

$$\sqrt{\frac{\omega\rho \sin\lambda}{\mu}}$$

we find the following relation between H_1 and the eddy viscosity

$$\frac{\mu}{\rho} = \frac{H_1{}^2 \omega \sin\lambda}{(2{\cdot}7)^2}.$$

But ω, the angular velocity of the earth, is $0{\cdot}000073$; and in latitude 50 degrees N., which is the latitude of the South of England and also of the northern portions of the Bank of Newfoundland, $\sin\lambda = 0{\cdot}77$.

Hence for those regions $\frac{\mu}{\rho} = H_1{}^2 \times 0{\cdot}77 \times 10^{-5}$.

On land, in the case of the strong winds,* $H_1 = 900$ metres, hence

$$\frac{\mu}{\rho} = 62 \times 10^3 \text{ in C.G.S. units};$$

for moderate winds,*

$$H_1 = 800 \text{ metres and } \mu/\rho = 50 \times 10^3;$$

and for light winds,*

$$H_1 = 600 \text{ metres and } \mu/\rho = 28 \times 10^3.$$

At sea,† in the regions to which the "Scotia's" cruises were confined, H_1 commonly lay between 100 metres and 300 metres so that μ/ρ lay between $0{\cdot}77 \times 10^3$ and $6{\cdot}9 \times 10^3$.

* See Mr. DOBSON'S paper, *loc. cit.*

† Assuming that the wind had reached the gradient velocity when it had practically stopped veering with increasing height.

Except for the kite ascent of July 17th, 1913, the values of $\kappa/\rho\sigma$ which, as was shown on p. 14, should be equal to μ/ρ, lie between these values.* It is unfortunate that the lack of skilled assistance in flying the kites from the "Scotia" prevented me in most cases from being able to get simultaneous values of $\kappa/\rho\sigma$ and μ/ρ. For the kite ascent of August 2nd, however, I have the following observations :—At 350 feet the wind had veered one point from the surface wind. At 770 feet the wind had veered two points from the surface wind, and at all greater heights the veer was two points. It seems, therefore, that at 770 feet, *i.e.*, 230 metres, or at some less height, the wind had attained the gradient direction, so the μ/ρ lay between $0·77 \times (23{,}000)^2 \times 10^{-5}$ or $4·0 \times 10^3$ and $0·77 \times 10^3$. On referring to Table I. it will be seen that the value of $\kappa/\rho\sigma$ on that occasion was $2·5 \times 10^3$. These results certainly tend to confirm the theoretical deduction that $\kappa/\rho\sigma = \mu/\rho$, but more evidence is wanted before the point can be regarded as finally settled.

On p. 13 it was shown that $\mu/\rho = \frac{1}{2}(\overline{w}d)$. The size of the eddies, which produce the effects we have been considering, are evidently governed by d. We may say roughly that d is less than the average diameter of an eddy; if therefore we could measure $\overline{w}$, we should be able to determine the size of the eddies. Now Mr. J. S. DINES has made a large number of observations of small vertical gusts with tethered balloons. On p. 216 of the Technical Report of the Advisory Committee for Aeronautics is shown a trace which represents the vertical component of the wind velocity at any time during a certain interval of five minutes, on January 19th, 1912. The average wind velocity during the interval was 7 metres per second; and I find from the trace, which Mr. DINES says is typical, that the average deviation from the mean vertical velocity (the mean wind was not quite horizontal) was 25 cm. per second. We may take this as $\overline{w}$. Assuming that the gradient direction was attained at a height of 800 metres the value of $\frac{1}{2}(\overline{w}d)$ would be 50×10^3 or $\overline{w}d = 10^5$ approximately.

Hence

$$d = \frac{10^5}{25} = 4 \times 10^3 \text{ cm.} = 40 \text{ metres.}$$

The wind was blowing with velocity 7 metres per second so that it would cover $7 \times 60 = 420$ metres, or about 10 times d, in a minute. If the vertical and horizontal dimensions of an eddy are about the same, this would mean (since d is less than the diameter of an eddy) that rather less than 10 eddies would pass a given spot in a minute. On examining Mr. DINES' trace it will be found that there are roughly about 6 peaks per minute on the curve representing vertical velocity.

These calculations are very rough, but they do show at any rate, that actual observations of eddy motion do not involve anything that is contrary to the assumptions on which the theory contained in this paper is based.

* See Table I.

NOTE ON THE STABILITY OF LAMINAR MOTION OF AN INVISCID FLUID, MAY 26TH.

The equation (8) throws a new light on the much discussed question of the stability of the laminar motion of an inviscid fluid.

Lord RAYLEIGH has considered the stability of a fluid moving in such a way that U, the undisturbed velocity, is parallel to the axis of x and is a function of z. His method is to impose a small disturbing velocity of a type which is simple harmonic with respect to x, satisfies the equations of motion, and contains a factor e^{int}. He then discusses the conditions under which n may be complex. If n is not complex the motion is stable; if n is complex the motion is exponentially unstable.

Perhaps the most important result of Lord RAYLEIGH'S investigation is the conclusion he arrives at that if $d^2\mathrm{U}/dz^2$ does not change sign in the space between any two bounding planes, unstable motion is impossible. A particular case of laminar motion in which $d^2\mathrm{U}/dz^2$ has the same sign throughout the fluid is that of an inviscid fluid flowing with the same velocity as a viscous liquid moving under pressure between two parallel planes. In this case, therefore, unstable motion should be impossible. OSBORNE REYNOLDS, however, working in an entirely different way, has come to the conclusion that a viscous fluid moving between parallel planes is unstable if the coefficient of viscosity is less than a certain value which depends on the distance between the planes and on the velocity of the fluid. REYNOLD'S result is in accordance with our experimental knowledge of the behaviour of actual fluids.

It is evident that there is a fundamental disagreement between the two results for, according to REYNOLDS, the more nearly inviscid the fluid, the more unstable it is likely to be; while according to RAYLEIGH instability is impossible when the fluid is quite inviscid.

Various attempts have been made to find the cause of the disagreement, but none of them appear to have been very successful.

The object of this note is to show that equation (8) may be used to prove the truth of Lord RAYLEIGH'S result for the case of a general disturbance, not necessarily harmonic with respect to x; and to show also that it may be used to assign a reason for the difference between RAYLEIGH'S and REYNOLDS' results.

Starting from the principle that when an inviscid fluid in laminar motion is disturbed by dynamical instability, each portion of it retains the vorticity of the layer from which it started, it was shown* that the rate at which momentum parallel to the axis of x flows into a slab of area A and thickness δz is :—

$$-\rho\left[\frac{d^2\mathrm{U}}{dz^2}\iint_{\mathrm{A}} w\,(z_0-z)\,dx\,dy+\frac{1}{2\,!}\frac{d^3\mathrm{U}}{dz^3}\iint_{\mathrm{A}} w\,(z_0-z)^2\,dx\,dy+\,\ldots\right]\delta z,$$

the integrals being taken over the area A.

* See p. 13.

This expression is true for all disturbances, however large, but when the distance $z_0 - z$ is small the first term only is of importance. Now it is evident that w is related to $z_0 - z$ by the relations

$$\frac{\mathrm{D}}{\mathrm{D}t}(z_0) = 0, \qquad \frac{\mathrm{D}}{\mathrm{D}t}(z) = w.$$

Hence

$$\frac{\mathrm{D}}{\mathrm{D}t}(z_0 - z) = -w,$$

or

$$(\mathrm{U}+u)\frac{\partial}{\partial x}(z_0-z) + w\frac{\partial}{\partial z}(z_0-z) + \frac{\partial}{\partial t}(z_0-z) = -w.$$

If u, w, and z_0-z are small, this is equivalent to

$$-w = \frac{\partial}{\partial t}(z_0-z) + \mathrm{U}\frac{\partial}{\partial x}(z_0-z).$$

Hence

$$\iint_{\mathrm{A}} w\,(z_0-z)\,dx\,dy = -\,\mathrm{U}\iint_{\mathrm{A}} \frac{\partial}{\partial x}(z_0-z)\,dx\,dy - \iint_{\mathrm{A}} (z_0-z)\frac{\partial}{\partial t}(z_0-z)\,dx\,dy.$$

Now when a large area is considered $\iint_{\mathrm{A}} \frac{\partial}{\partial x}(z_0-z)\,dx\,dy$ integrates out and vanishes.

Hence

$$\iint_{\mathrm{A}} w\,(z_0-z)\,dx\,dy = -\iint_{\mathrm{A}} \tfrac{1}{2}\frac{\partial}{\partial t}(z_0-z)^2\,dx\,dy = -\,\tfrac{1}{2}\frac{d}{dt}\iint_{\mathrm{A}}(z_0-z)^2\,dx\,dy.$$

It appears therefore that the rate at which the x-momentum in the slab A increases is

$$\tfrac{1}{2}\rho\frac{d^2\mathrm{U}}{dz^2}\,\delta z\,\frac{d}{dt}\left[\iint_{\mathrm{A}}(z_0-z)^2\,dx\,dy\right].$$

Integrating with respect to t we find that the difference between the momentum in the slab A after and before the disturbance set in is

$$\tfrac{1}{2}\rho\,\frac{d^2\mathrm{U}}{dz^2}\,\delta z\iint_{\mathrm{A}}(z_0-z)^2\,dx\,dy.$$

Lord RAYLEIGH has pointed out that it is difficult to define instability. In the present case the motion will be held to be unstable if the average value of the square of the distance of any portion of the fluid from the layer out of which the disturbance has removed it, increases with time. This evidently includes the case of exponentially unstable simple harmonic waves.

In unstable motion therefore $\frac{d}{dt}\iint_{\mathrm{A}} (z_0-z)^2\, dx\, dy$ must be positive.

Hence the rate at which x-momentum enters the slab A is positive or negative according as $d^2\mathrm{U}/dz^2$ is positive or negative. In an unstable disturbance of a fluid for which $d^2\mathrm{U}/dz^2$ is everywhere positive the momentum of every layer must increase. But if there is perfect slipping at the boundaries no momentum can be communicated by them. Hence, as there is no other possible source from which the momentum can be derived, instability cannot possibly occur. The argument applies equally well if $d^2\mathrm{U}/dz^2$ is everywhere negative. Lord RAYLEIGH'S result is therefore proved for a generalised disturbance. In a case where $d^2\mathrm{U}/dz^2$ changes sign at some point in the fluid any disturbance reduces the x-momentum in a layer where $d^2\mathrm{U}/dz^2$ is negative, while it increases the x-momentum in layers in which $d^2\mathrm{U}/dz^2$ is positive. A type of disturbance which removes x-momentum from places where $d^2\mathrm{U}/dz^2$ is negative and replaces it in regions where $d^2\mathrm{U}/dz^2$ is positive, so that there is no necessity for the boundaries to contribute, may be unstable.

Now consider what modifications must be made in the conditions in order that instability may be possible in the case where $d^2\mathrm{U}/dz^2$ is of the same sign throughout (say negative). Suppose that instability is set up so that x-momentum flows outwards from the central regions as the disturbance increases. The amount of x-momentum crossing outwards towards the walls through a plane perpendicular to the axis of z, increases as the walls are approached. In order that instability may be set up this momentum must be absorbed by the walls. There seems to be no particular reason why an infinitesimal amount of viscosity should not cause a finite amount of momentum to be absorbed by the walls.

In connection with this two points should be noticed. Firstly, the momentum is only communicated to the walls while the disturbance is being produced. The time necessary to produce a given disturbance may increase as the viscosity diminishes. Experimental evidence, however, does not suggest that this is the case.

The second point is suggested by the conclusion arrived at on pp. 11–22, that a very large amount of momentum is communicated by means of eddies from the atmosphere to the ground. This momentum must ultimately pass from the eddies to the ground by means of the almost infinitesimal viscosity of the air. The actual value of the viscosity of the air does not affect the rate at which momentum is communicated to the ground, although it is the agent by means of which the transference is effected.

In any case it is obvious that there is a finite difference, in regard to slipping at the walls, between a perfectly inviscid fluid and one which has an infinitestimal viscosity. The distribution of velocity acquired by a viscous fluid flowing between parallel planes at which there is no slipping is possible for an inviscid fluid when there is perfect slipping, but is impossible as a steady state for an infinitesimally viscous fluid which slips at the boundaries.

The finite loss of momentum at the walls due to an infinitesimal viscosity may be compared with the finite loss of energy due to an infinitesimal viscosity at a surface of discontinuity in a gas.*

If these views are correct we should expect that Lord RAYLEIGH's result would not apply when there are no bounding planes and space is filled with a fluid in which d^2U/dz^2 is everywhere positive; for, in that case, there would be nothing to prevent a positive amount of x-momentum from being communicated to every portion of the fluid, provided the disturbance increases indefinitely for infinitely great values of z. In obtaining his result Lord RAYLEIGH assumes that, if there are no bounding planes, $w = 0$ at infinity;† it does not apply therefore to the case just considered.

The conclusion arrived at is that the discrepancy between RAYLEIGH's and REYNOLDS' results is due to the fact that the perfect slipping at the boundaries assumed in RAYLEIGH's work prevents the escape of the momentum which is a necessary accompaniment of a disturbance of a fluid for which d^2U/dz^2 is everywhere negative. The complete absence of slipping assumed in REYNOLDS' work enables the necessary amount of momentum to escape, and so a type of disturbance may be produced which is dynamically impossible under the condition of perfect slipping at the boundaries.

* See "Conditions Necessary for Discontinuous Motion in Gases," TAYLOR, 'Roy. Soc. Proc.,' 1910, A, vol. 84, p. 371.

† 'Phil. Mag.,' vol. 26, 1913, p. 1002.