Long Waves and Cyclone Waves

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Abstract

By obtaining complete solutions, satisfying all the relevant simultaneous differential equations and boundary conditions, representing small disturbances of simple states of steady baroclinic large-scale atmospheric motion it is shown that these simple states of motion are almost invariably unstable. An arbitrary disturbance (corresponding to some inhomogeneity of an actual system) may be regarded as analysed into "components" of a certain simple type, some of which grow exponentially with time. In all the cases examined there exists one particular component which grows faster than any other. It is shown how, by a process analogous to "natural selection", this component becomes dominant in that almost any disturbance tends eventually to a definite size, structure and growth-rate (and to a characteristic life-history after the disturbance has ceased to be "small"), which depends only on the broad characteristics of the initial (unperturbed) system. The characteristic disturbances (forms of breakdown) of certain types of initial system (approximating to those observed in practice) are identified as the ideal forms of the observed cyclone waves and long waves of middle and high latitudes. The implications regarding the ultimate limitations of weather forecasting are discussed.

The present paper aims at developing from first principles a quantitative theory of the initial stages of development of wave-cyclones and long waves. For reasons of space and readability both the argument and the mathematics have been rather heavily compressed. A fuller and extended treatment of several of the points raised will be given in subsequent papers.

I. The Equations of Motion

Owing to the complexity (and non-linearity) of the simultaneous partial differential equations governing atmospheric motion it is desirable to simplify these by the omission of all those terms which do not make a major contribution *to the particular type and scale of motion envisaged*. This procedure is made possible by the fact that we know, from observation, roughly what the answers must look like. Its utility is justified by the fact that, having once obtained a crude model of the motion, we may then by successive approximation take into account any or all of the terms originally omitted. In the present instance we are concerned with relatively rapid development, by comparison with which radiative processes (or rather their differential effects) are slow. For a first approximation therefore we consider the motion as adiabatic. Also we are concerned with the motion of deep layers and for a first approximation we neglect the effects of internal friction ("turbulence") and skin friction. A rough calculation shows that the energy dissipated in the surface friction layer is usually much less than the energy supply to the growing disturbance and this is probably, in most cases, the major source of energy loss. The present paper is concerned only with systems which are initially (and also after a small perturbation) convectively stable, i.e., with those systems which would appear to be least favourable with regard to instability. Hence we use a system of equations appropriate to laminar frictionless adiabatic motion of a rotating baroclinic fluid. Restriction to convective stability makes possible a further slight simplification in that in almost all[1] such cases (except very close to the border-line) we may neglect vertical accelerations and use the hydrostatic equation — the disturbances are "quasi-static". Briefly, the explanation is that in such cases the energy associated with horizontal perturbations greatly exceeds that associated with vertical motion.

The atmosphere is a compressible fluid and in estimating the significance of this fact it is convenient to consider separately the static effect, manifested by the decrease of density with height, and the dynamic effect, manifested by elastic forces in the equations of motion. The static effect has two consequences. In the first place, the static stability, a measure of the force tending to restore a displaced particle to its equilibrium position, is measured not by the vertical density gradient but by the gradient of potential density (or by the difference between actual and adiabatic lapse-rate). As compared with incompressible flow this involves only the modification of a parameter. In the second place, a given mass of air occupies, at higher levels, a greater height range. The result is that atmospheric flow can never be quite equivalent to incompressible flow but (as may be inferred from the detailed treatment) the difference is to be regarded as a distortion rather than any difference in kind. For systems which are not too deep there is an equivalent incompressible system whose behaviour closely parallels that of the atmospheric one. The nature of the "correction" for very deep systems is discussed below. The significance of the elastic forces depends on the type of solution in which we are interested. In the theory of atmospheric tides and the diurnal variation of pressure, where the wave-velocities are comparable with the speed of sound, these forces play an essential part. But in all waves associated with "weather" the wave-velocities are, by observation, small compared with the speed of sound and this is true, as we shall see, of both the real and imaginary parts of the wave-velocities of the theoretical solutions. We are therefore justified in treating the motion, from the dynamic point of view, as incompressible. The net result is that we can construct an equivalent "incompressible flow" problem and then use as our continuity equation:[1]

$$\mathrm{div}_H \mathbf{v} + \frac{\partial}{\partial z} V_z = 0 \qquad (1)$$

For systems which are not too large we may use the ordinary cartesian co-ordinates fixed in the earth in which we imagine a small spherical cap to be "flattened" on to the tangent plane so that gravity acts along parallel lines. This involves a certain amount of geometrical distortion (and consequently a distortion of our solutions). A more serious error results from the assumption of a constant Coriolis parameter and we may obtain a first approximation to this error by using the same co-ordinate system but regarding the Coriolis parameter as a function of y (the N—S co-ordinate). A more precise treatment of long waves requires the use of a polar (or equivalent) co-ordinate system and crude solutions, using numerical methods, have been obtained in this case. The broad resemblance of these solutions to those obtained by analytical methods in the cartesian system justifies the use of the latter as a first rough approximation.

If we define, for unsaturated air:

$$\Phi = \frac{1}{\gamma} \log p - \log \varrho \qquad (2)$$

so that Φ is proportional to the entropy, we have for adiabatic motion:

$$\frac{d}{dt} \Phi = 0 \qquad (3)$$

We shall *define* the static stability as $\frac{\partial}{\partial z} \Phi$. To study the motion of saturated air in contact with cloud we have only to alter the effective value of the static stability (our norm is now the wet adiabatic). Usually the reduction is

[1] The exception is the case of strong anticyclonic horizontal shear approaching in magnitude the Coriolis parameter—see below.

[1] When not otherwise stated, the symbols employed are those normally used in theoretical meteorological literature.

very significant, the forces opposing vertical motion being much less inside than outside a cloud, a fact having, as we shall see, important consequences. Moreover we obtain a direct translation from atmospheric motion to incompressible flow by the substitution:

$$\Phi \to -\log \varrho \qquad (4)$$

Alternatively, we may regard γ as having, for saturated air, a value slowly varying from considerably less than 1.4 at high temperatures but asymptotically approaching the dry adiabatic value at low temperatures. For an incompressible fluid γ is effectively infinite.

With the approximations mentioned above we have the dynamic equations (where K is the Coriolis parameter):

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = \frac{d}{dt}V_x - KV_y$$

$$-\frac{1}{\varrho}\frac{\partial p}{\partial y} = \frac{d}{dt}V_y + KV_x$$

$$-\frac{1}{\varrho}\frac{\partial p}{\partial z} = g \qquad (5)$$

These equations, together with equations (I. 1)—(I. 3), form a complete set. Further simplification is however possible. (For the present we take K to be constant.) By partial differentiation with respect to z of the first two of equations (I. 5), using equation (I. 2) and the last of equations (I. 5):

$$\left(\frac{\partial}{\partial z} - \frac{\partial \Phi}{\partial z}\right)\left(\frac{d}{dt}V_x - KV_y\right) = -g\frac{\partial \Phi}{\partial x},$$

$$\left(\frac{\partial}{\partial z} - \frac{\partial \Phi}{\partial z}\right)\left(\frac{d}{dt}V_y + KV_x\right) = -g\frac{\partial \Phi}{\partial y}, \qquad (6)$$

where the first brackets are regarded as operators. We shall see that the solutions with which we are concerned "oscillate" in the direction of z (it is immaterial that the "oscillation" is not sinusoidal) and if this "oscillation" is sufficiently rapid (if the disturbances are not too deep) the effect of the operator $\frac{\partial}{\partial z}$ swamps that of the multiplier $\frac{\partial \Phi}{\partial z}$. By numerical substitution in the final answers we can verify that this is the case for the solutions in which we are interested. Then for a first approximation we may use in place of (I. 6):

$$\frac{\partial}{\partial z}\left(\frac{d}{dt}V_x - KV_y\right) = -g\frac{\partial \Phi}{\partial x},$$

$$\frac{\partial}{\partial z}\left(\frac{d}{dt}V_y + KV_x\right) = -g\frac{\partial \Phi}{\partial y}. \qquad (7)$$

Equations (I. 7) together with (I. 1) and (I. 3) form a complete set involving only the dependent variables V_x, V_y, V_z, Φ (note that the operator $\frac{d}{dt}$ involves V_x, V_y, V_z: the equations are non-linear). However in the elimination process we have lost a function of integration. If we differentiate the first two of equations (I. 5) with respect to y and x respectively and subtract we obtain, using (I. 2):

$$\left(\frac{\partial}{\partial y} - \frac{\partial \Phi}{\partial y}\right)\left(\frac{d}{dt}V_x - KV_y\right) -$$

$$-\left(\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x}\right)\left(\frac{d}{dt}V_y + KV_x\right) = 0 \qquad (8)$$

Numerical substitution then shows that in the cases in which we are interested the effects of the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ swamp those of the multipliers $\frac{\partial \Phi}{\partial x}$, $\frac{\partial \Phi}{\partial y}$ and to a sufficiently close approximation:

$$\frac{\partial}{\partial y}\left(\frac{d}{dt}V_x - KV_y\right) - \frac{\partial}{\partial x}\left(\frac{d}{dt}V_y + KV_x\right) = 0 \qquad (9)$$

Elimination of Φ from equations (I. 7) gives simply $\frac{\partial}{\partial z}$ of equation (I. 9) — the approximations in the two cases are consistent. On rearrangement of terms (I. 9) becomes:

$$\mathrm{div}_H\mathbf{v}\cdot(K + \mathrm{curl}_H\mathbf{v}) + \frac{d}{dt}\mathrm{curl}_H\mathbf{v} +$$

$$+\left(\frac{\partial V_z}{\partial x}\cdot\frac{\partial V_y}{\partial z} - \frac{\partial V_z}{\partial y}\cdot\frac{\partial V_x}{\partial z}\right) = 0 \qquad (10)$$

To obtain a fourth equation symmetrical in V_x and V_y we differentiate equations (I. 7) with respect to x and y respectively and add. Then on re-arrangement:

$$\frac{\partial}{\partial z}\left[K\,\mathrm{curl}_H\mathbf{v}-\frac{d}{dt}\,\mathrm{div}_H\mathbf{v}-\left\{\left(\frac{\partial V_x}{\partial x}\right)^2+\right.\right.$$
$$\left.+2\,\frac{\partial V_x}{\partial y}\cdot\frac{\partial V_y}{\partial x}+\left(\frac{\partial V_y}{\partial y}\right)^2\right\}-\left(\frac{\partial V_z}{\partial x}\cdot\frac{\partial V_x}{\partial z}+\right.$$
$$\left.\left.+\frac{\partial V_z}{\partial y}\cdot\frac{\partial V_y}{\partial z}\right)\right]=g\cdot\nabla_H^2\Phi \qquad (11)$$

where the suffix H in all cases indicates differentiation with respect to x, y only. We shall use equations (I. 1), (I. 3), (I. 10) and (I. 11) as our fundamental set. When other dependent variables are required (e.g., pressure, which may appear in boundary conditions) they are easily computed in terms of our fundamental set.

An important feature of this set of equations is that transformation to a co-ordinate system in uniform horizontal relative motion is almost as simple as in the Newtonian case. V_x and V_y transform as in the latter case by vectorial addition of the relative velocity, V_z and Φ being unchanged. The only difference is that we must add a pressure field (at all levels) whose gradient corresponds to the relative velocity regarded as a geostrophic wind. This is consistent with the assumption that the so-called "tendency equation" is not to be interpreted as an expression of accumulation of mass but rather that the terms associated with this process are usually negligibly small compared with those associated with change of flow — as in classical (subsonic) aerodynamics.

The approximations made could be more convincingly justified by a detailed analysis, but a rigorous proof is possible only *after* obtaining the complete solutions of the approximate equations, when we can check on the precise effect of the omitted terms. A simpler but nevertheless fairly convincing check is the "life-like" behaviour of the solutions, both qualitatively and quantitatively.

II. Disturbances of Steady Baroclinic Flow

We shall consider first a state of steady baroclinic flow in which the motion is uniform at each level and for simplicity we shall suppose both the "thermal wind" and the static stability constant, i.e., Φ is a linear function of x, y, z. In view of the transformation theorem referred to above there is no loss of generality in supposing:

$$\begin{aligned} V_x &= U(z): \\ V_y &= V_z = 0: \\ \Phi &= Ay + Bz: \end{aligned} \qquad (1)$$

So long as we ignore the variability of K the equations of motion are horizontally isotropic — the orientation of the y-axis is irrelevant. Equations (II. 1) are consistent with steady motion if:

$$\frac{dU}{dz}=-\frac{gA}{K} \qquad (2)$$

by reason of (I. 6). The approximation is good except when the thermal wind is very small (or the pressure gradient abnormally large). Since we are interested in the behaviour of strongly baroclinic systems no serious errors are introduced.

We now introduce a small perturbation and write:

$$\begin{aligned} V_x &= U + v_x: \\ V_y &= v_y: \\ V_z &= v_z: \\ \Phi &= Ay + Bz + \varphi: \end{aligned} \qquad (3)$$

where v_x, v_y, v_z, φ are infinitesimal functions of x, y, z, t. Substituting in our fundamental set of equations we obtain the perturbation equations (which are of course linear)

$$\mathrm{div}_H\mathbf{v}+\frac{\partial}{\partial z}v_z=0:$$

$$K^2(av_y+bv_z)+\frac{d}{dt}(g\varphi)=0:$$

$$K\,\mathrm{div}_H\mathbf{v}+\frac{d}{dt}\mathrm{curl}_H\mathbf{v}+Ka\frac{\partial}{\partial y}v_z=0:$$

$$\frac{\partial}{\partial z}\left[K\,\mathrm{curl}_H\mathbf{v}-\frac{d}{dt}\,\mathrm{div}_H\mathbf{v}+\right.$$
$$\left.+Ka\frac{\partial v_z}{\partial x}\right]=\nabla_H^2(g\varphi), \qquad (4)$$

where:

$$a \equiv \frac{gA}{K^2}\,;\; b \equiv \frac{gB}{K^2}\,;\; \frac{d}{dt} = U\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\,; \quad (5)$$

By using the identities:

$$\left(\frac{\partial}{\partial z}\frac{d}{dt} - \frac{d}{dt}\frac{\partial}{\partial z}\right) \equiv -Ka\frac{\partial}{\partial x}:$$

$$\nabla_H^2 v_y \equiv \frac{\partial}{\partial y}\,\mathrm{div}_H\mathbf{v} + \frac{\partial}{\partial x}\,\mathrm{curl}_H\mathbf{v}: \quad (6)$$

we may eliminate successively φ, $\mathrm{curl}_H\mathbf{v}$, $\mathrm{div}_H\mathbf{v}$ from equations (II. 4). Finally we obtain a single partial differential equation involving v_z as the only dependent variable:

$$\frac{d}{dt}\left(K^2 + \frac{d^2}{dt^2}\right)\frac{\partial^2 v_z}{\partial z^2} + 2K^2 a\left(K\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial y}\right)\frac{\partial v_z}{\partial z} + K^2\left[b\frac{d}{dt}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2Ka^2\frac{\partial^2}{\partial x \partial y}\right]v_z = 0: \quad (7)$$

We cannot hope to solve this equation (technically of the fifth order) in complete generality but we may seek certain simple types of solution. Fortunately the simplest solutions are those of greatest practical importance. Since, apart from constants and differential operators (II. 7) involves only functions of z $\left(\text{the function } U(z) \text{ appearing in } \frac{d}{dt}\right)$ it clearly possesses solutions of the form:

$$v_z = N(z)\cdot\Psi : \Psi \equiv e^{i(\lambda x + \mu y + \vartheta t)}\,; \quad (8)$$

where λ, μ, ϑ are constants and N is a function of z only. In fact we may replace the operators:

$$\frac{\partial}{\partial x} = i\lambda\,;\; \frac{\partial}{\partial y} = i\mu\,;\; \frac{d}{dt} = i(U\lambda + \vartheta)\,; \quad (9)$$

and if at the same time we change our vertical co-ordinate by writing:

$$X \equiv X(z) = \frac{U\lambda + \vartheta}{K}\,;\; \frac{dX}{dz} = -a\lambda\,; \quad (10)$$

we obtain on substitution the ordinary second order differential equation to determine N:

$$X(X^2 - 1)\frac{d^2N}{dX^2} + 2(1 - i\sigma X)\frac{dN}{dX} + [h^2(1+\sigma^2)X + 2i\sigma]N = 0 \quad (11)$$

where we have written:

$$h^2 \equiv \frac{b}{a^2} = \frac{gB}{\left(\frac{dU}{dz}\right)^2}:$$

$$\sigma \equiv \frac{\mu}{\lambda}: \quad (12)$$

The parameter h^2, which involves both the horizontal and vertical entropy gradients, sums up (apart from matters of scale and boundary conditions) the characteristic properties of the flow. From (I. 4) it is clear that h^2 is simply the Richardson number of the unperturbed flow.

If we can solve (II. 11) with appropriate boundary conditions all the associated perturbation functions are readily determined by means of equations (II. 4). In fact a consistent set of solutions may be obtained in which:

$$v_x = L(z)\cdot\Psi:$$
$$v_y = M(z)\cdot\Psi:$$
$$\varphi = F(z)\cdot\Psi:$$
$$p = G(z)\cdot\Psi, \quad (13)$$

where the pressure perturbation p is determined by:

$$-\frac{1}{\varrho_0}\frac{\partial p}{\partial x} = \frac{d}{dt}v_x - Kv_y - Kav_z \quad (14)$$

or the corresponding equation for $\frac{\partial p}{\partial y}$. Here ϱ_0 is quite uncritical and we may take $\varrho_0 = \varrho_0(z)$ as the mean distribution of density with height in the system with which we are concerned.

For solutions of the form (II. 8) and (II. 13) we obtain the relations:

$$\frac{(1+\sigma^2)}{a}\cdot M = \frac{1}{X}\frac{dN}{dX} + i\sigma\left(\frac{N}{X} - \frac{dN}{dX}\right);$$

$$\frac{(1+\sigma^2)}{a}\cdot L = -\frac{\sigma}{X}\frac{dN}{dX} - i\left(\sigma^2\frac{N}{X} + \frac{dN}{dX}\right);$$

$$\frac{(1+\sigma^2)}{a}\cdot\frac{g}{Ka}\cdot F = \frac{i}{X}\left[\left\{\frac{1}{X}\frac{dN}{dX} + h^2(1+\sigma^2)N\right\} + \right.$$

$$\left. + i\sigma\left\{\frac{N}{X} - \frac{dN}{dX}\right\}\right];$$

$$\frac{(1+\sigma^2)}{a}\cdot\frac{i\lambda}{K\varrho_0}\cdot G =$$

$$= \left(\frac{1}{X}\frac{dN}{dX} + i\sigma\frac{N}{X}\right) + \left(N - X\frac{dN}{dX}\right) \quad (15)$$

With the substitution:

$$N = \left(\frac{1-X}{1+X}\right)^{\frac{i\sigma}{2}}\cdot R \quad (16)$$

equation (II. 11) becomes

$$X(X^2-1)\frac{d^2R}{dX^2} + 2\frac{dR}{dX} + X\left[h^2(1+\sigma^2) + \right.$$

$$\left. + \frac{\sigma^2}{(X^2-1)}\right]R = 0 \quad (17)$$

which is in some ways more convenient.

We will suppose for the moment that λ, μ are real. If at the same time ϑ is real the solution (II. 8) will correspond to a system of stable waves. For the unstable waves we are seeking, ϑ must have a non-vanishing imaginary part and we shall write:

$$\vartheta = \vartheta_0 - i\,\vartheta_1 \quad (18)$$

For such solutions (if any) X defined by (II. 10) becomes a complex variable (with a constant imaginary part) and it is convenient to regard R, determined by (II. 17), as a function of this complex variable. Thus in general L, M, N etc. as well as Ψ are complex numbers. This in no way affects the physical interpretation of v_x, v_y, v_z etc., as the real parts (for example) of the expressions (II. 8) and (II. 13). All it means is that the phase of the wave corresponding to each element (velocity-component, pressure etc.) as well as the amplitude varies with height.

Numerical substitution shows that in normal, convectively stable conditions (we are concerned with mean values over considerable depths) we have:

$$h^2 \gg 1 \quad (19)$$

We shall for the present confine our attention to this, the most interesting case. It can be shown by energy considerations (see below) that in this case we should have:

$$|X^2| \ll 1 \quad (20)$$

over the range for which the perturbations have significant amplitudes. Alternatively we may assume this result and show that our final solutions are consistent with this assumption. Then (II. 17) becomes approximately:

$$\frac{d^2R}{dX^2} - \frac{2}{X}\frac{dR}{dX} - h^2(1+\sigma^2)R = 0 \quad (21)$$

and with the substitutions:

$$H^2 = h^2(1+\sigma^2);\ Q = HX; \quad (22)$$

we obtain:

$$\frac{d^2R}{dQ^2} - \frac{2}{Q}\frac{dR}{dQ} - R = 0 \quad (23)$$

the general solution of which is:

$$R = a_1R_1 + a_2R_2$$
$$R_1 \equiv e^{Q}(1-Q)$$
$$R_2 \equiv e^{-Q}(1+Q): \quad (24)$$

where a_1, a_2 are arbitrary constants.

Consider first a hypothetical (but physically possible) system in which motion takes place between two horizontal rigid plane boundaries which, without loss of generality, we may suppose to be at $z = \pm\frac{z_0}{2}$ corresponding to X_1, Q_1, and X_2, Q_2 respectively. We shall suppose the fluid unbounded in any horizontal direction. Then for the functions to be finite at infinity we must take λ, μ real. At $z = \pm \frac{z_0}{2}$ the normal velocity v_z must vanish which will be the case if:

$$R = 0;\ Q = Q_1,\ Q_2 \qquad (25)$$

Then we have:

$$-\frac{a_2}{a_1} = e^{2Q_1}\left(\frac{1 - Q_1}{1 + Q_1}\right) = e^{2Q_2}\left(\frac{1 - Q_2}{1 + Q_2}\right): \qquad (26)$$

and if we write:

$$(Q_2 - Q_1) = 2\,\alpha:$$
$$(Q_2 + Q_1) = -2\,i\beta, \qquad (27)$$

we have:

$$\beta^2 = 2\,\alpha \coth 2\,\alpha - 1 - \alpha^2 \equiv$$
$$\equiv (\alpha - \tanh \alpha)(\coth \alpha - \alpha) \qquad (28)$$

as the condition to be satisfied if solutions are to exist. On substitution for Q_1, Q_2 in (II.27) we have:

$$2\,\alpha = \frac{H}{K} \cdot \frac{dU}{dz}\lambda z_0:$$
$$-2\,i\beta = \frac{H}{K}\left[\lambda\left(U_1 + U_2\right) + 2\,\vartheta_0\right]$$
$$-2\,i\vartheta_1 \cdot \frac{H}{K} \qquad (29)$$

so that α is necessarily real. It follows from (II. 28) that β is either purely real or purely imaginary. Only when β is real, i.e. when

$$|\alpha| < \alpha_0 = 1.1997\ [\alpha_0 = \coth \alpha_0] \qquad (30)$$

do unstable solutions exist. Then by (II. 29):

$$\vartheta_1 = \beta \cdot \frac{K}{H}: -\frac{\vartheta_0}{\lambda} = \frac{(U_1 + U_2)}{2} \qquad (31)$$

where U_1, U_2 are the unperturbed velocities at levels $z = \pm \frac{z_0}{2}$ respectively. The second of these equations shows that the waves travel with the mean unperturbed current (we may

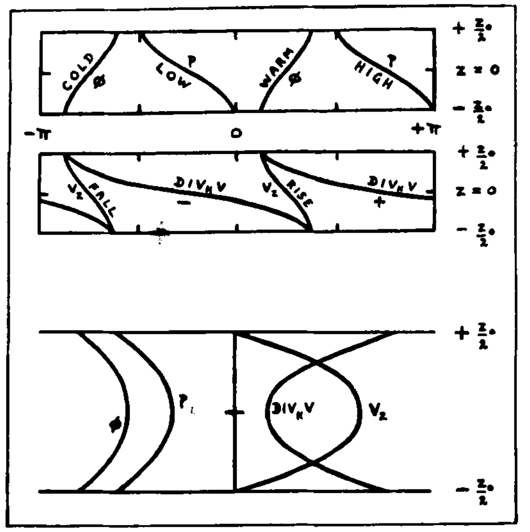

Fig. 1 a. Phase Variations: Above p, φ; Below v_z, $\text{div}_H\mathbf{v}$;
Fig. 1 b. Amplitude Variations: Left p, φ; Right v_z, $\text{div}_H\mathbf{v}$.

call the level $z = 0$ the "steering level"). The first equation determines the growth-rate. It is easily shown that $|\beta|$ defined by (II. 28) has a maximum for a particular value of α. Then:

$$|\beta| = 0.3098$$
$$|\alpha| = 0.8031 \qquad (32)$$

For these values, with the additional condition $\sigma = 0$, the growth-rate is a maximum. At the same time (II. 29) determines the wavelength $\frac{2\pi}{\lambda}$ in terms of the parameters of the unperturbed system. The disturbances are easily seen to be, at each level, a series of (growing) ridges and troughs with their axes at right angles to the unperturbed thermal wind. Since the structure of these disturbances is very similar to that of the disturbances of more realistic systems it is of interest to examine them in detail. This structure is most conveniently described by graphs showing the variation with height of the phase and amplitude of the waves representing various salient features such as the pressure perturbation, vertical velocity, etc. The graphs for p, φ, v_z, $\text{div}_H\mathbf{v}$ are shown in Fig. 1. The distribution for v_y may be inferred directly from the pressure field since it is easily shown that for these disturbances [and in fact for the disturbances of all systems satisfying (II. 19)] the winds are to a first approximation geostrophic. Ageostrophic winds (including in the present instance v_x) are of order $\frac{1}{h}$.

It will be observed that the pressure trough slopes upwards and backwards in the atmosphere while the warm tongue (entropy maximum) slopes upwards and forwards. At low levels the warm tongue is slightly ahead of the pressure trough but at high levels the warm tongue is slightly to the rear of the upper pressure ridge. Upward motion (and hence, potentially, rainfall) is a maximum at middle levels $^1/_8$ wavelength ahead of the surface pressure trough. We may note that at middle levels v_y and v_z are in phase, the combined motion being upwards towards cold, downwards towards warm air. We may note also that at the same level v_z and φ are in phase with rising warm and descending cold air, corresponding to a decrease in potential energy. In fact on integration over a whole wavelength we find that there is a positive correlation between v_z and φ and a net decrease in the potential energy of the system as a result of the disturbance. It is of course this release of potential energy which feeds the kinetic energy of the growing disturbance. Our analysis shows that such a process (similar to that conceived by Margules) is consistent with all the equations and constraints of motion and in fact that such processes must occur from time to time.

It is easily verified by substituting typical values of the parameters that if z_0 is taken as the height of the tropopause the wavelength of the disturbance of maximum growth-rate is of approximately the same size as observed long waves. For smaller values of the static stability (as in large cloud masses) and smaller vertical extents we obtain "dominant" wavelengths of the order of magnitude of observed extratropical wave-cyclones. Thus we are certainly concerned with disturbances of the right order of magnitude. Our systems are not yet however sufficiently realistic for positive identification. As a first step towards realism we remove the artificial boundary at $z = +\frac{z_0}{2}$ and consider a system in which the atmosphere extends upwards indefinitely but at some definite level the static stability increases abruptly. For mathematical simplicity we take the "thermal wind" to be the same in both the lower and upper "regimes". This system is unsymmetrical so we shall put $z = 0$ at the earth's surface (rigid boundary) and $z = z_0$ at the boundary between the two regimes. Thus $z = z_0$ might correspond to an inversion or stabilisation of lapse-rate — were it not for the observed thermal wind change z_0 might be the height of the tropopause. Alternatively, the lower regime might be a (baroclinic) cloud mass of low base (small effective static stability) surmounted by unsaturated air. In all these cases the Richardson number (h_1^2) in the lower "regime" is less than that (h_2^2) in the upper regime. We can write down, as before, the general solution of (II. 21) appropriate to each regime. Clearly for continuity at the boundary we must have λ, μ, ϑ (and therefore X but not Q) the same in each regime. We still have two more arbitrary constants than before, but we have two additional "internal" boundary conditions since both the phase and amplitude (one complex number) of p and v_z (normal velocity) must be continuous at $z = z_0$. Our upper boundary condition is now $v_z \to 0$ as $z \to \infty$ for it is clear from (II. 24) that one of R_1, R_2 increases and the other decreases exponentially with height. (The boundary condition ensures that *all* the perturbation functions decrease exponentially.) Finally, we obtain, as before, a relation between λ and ϑ. If we write:

$$2\alpha = h_1 \sqrt{1 + \sigma^2}\, \frac{\lambda}{K} \frac{dU}{dz}\, z_0,$$

$$\beta = h_1 \sqrt{1 + \sigma^2} \cdot \frac{\vartheta_1}{K},$$

$$k_1 = \frac{h_1}{h_2}, \tag{33}$$

then

$$\beta^2 = \frac{(1 - k_1^2)\,(2\alpha - \tanh 2\alpha)}{(k_1 + \tanh 2\alpha)} - \left(\alpha - \frac{k_1}{2}\right)^2 \tag{34}$$

replaces (II. 28) while for the "steering level" as defined above:

$$z_\omega = \frac{z_0}{2}\left(1 + \frac{k_1}{2\alpha}\right) \qquad (35)$$

If we put $h_2 = \infty$ (infinite static stability) equations (II. 34) and (II. 35) reduce to formulae appropriate to the "two rigid boundary" system and the disturbance is confined to the lower regime. As k_1 increases from zero the disturbance gradually extends into the upper regime but as there is always exponential decrease with height in the upper regime, provided $k_1 < 1$, then, except near this limit, the actual conditions much above $z = z_0$ are quite uncritical. In many practical cases k_1 is nearer zero than unity and then the disturbance in the lower regime does not differ greatly from that of our original system (the limiting case). For a given value of k_1 we find as before that β is real only for sufficiently long waves (α sufficiently small) and for one particular wavelength β (and therefore growth-rate) is a maximum. As k_1 increases from zero β decreases slowly at first from its limiting value but vanishes when $k_1 = 1$. For $k_1 > 1$ there are no unstable solutions. (It would appear to be a general result that for instability of this type the Richardson number must have a minimum value within a certain finite region; the resulting disturbances then have their maximum amplitude in this region with exponential decrease in the surrounding regions. These conditions are satisfied in practice as a general rule.)

Similar calculations may be made for systems of three (or more) horizontally stratified regimes. An easily investigated system, representing the opposite extreme to the two-regime system discussed above, is that in which the static stability is relatively small within a layer which we take to be between levels $z = \pm\frac{z_0}{2}$ and relatively large both above and below. The outer regimes we take (for simplicity) to be of indefinite extent. This system corresponds, for example, to a baroclinic cloud mass of very high base. For simplicity we take $h = h_2$ in each of the outer regimes, $h = h_1$ in the inner regime. Then with the same notation as before:

$$\beta^2 = \frac{(1-k_1^2)\{2\alpha(\coth 2\alpha + k_1) - (1-k_1^2)\}}{(1 + k_1^2 + 2k_1 \coth 2\alpha)} - \alpha^2 \qquad (36)$$

The general behaviour is similar to that of the two-regime system except that now the disturbance decreases exponentially in both directions away from the inner regime. (For small values of k_1 it is almost as if the conditions at the boundaries of the inner regime were independent.) Thus disturbances developing on a high level cloud sheet, for example, would be unnoticed at ground level in their early stages.

Table 1 gives values of α, β corresponding to the "dominant" disturbance of maximum growth-rate calculated for $k_1^2 = 1/_6$, typical of the change from cloud to unsaturated air, for the three systems discussed above. Note the relatively small decrease in β due to "losses" at "imperfectly rigid" boundaries.

Table 1

System	I	II	III
α	0.8031	0.6190	0.4445
β	0.3098	0.2988	0.2885

The structure of the disturbances of the more complex systems may be studied in the same way as before. Figs. 2 and 3, to be interpreted as Fig. 1, correspond to the two-regime and three-regime systems respectively.

An interesting feature of the two-regime system appears when we compute the displacement of the internal boundary (the base of the change in lapse-rate) at $z = z_0$ due to the growing disturbance. This boundary (which might correspond roughly to the tropopause) is sucked down in the vicinity of the upper (i.e., at $z = z_0$) pressure minimum and pushed up in the vicinity of the upper pressure maximum (the phase coincidence becomes exact as $k_1 \to 0$, otherwise it is a close approximation). This result (opposite to that to be expected if cyclonic vorticity were generated by a simple vertical "stretching" — but note that behaviour in the upper regime, e.g., the stratosphere, *is* consistent with this view) is in good agreement with observed behaviour.

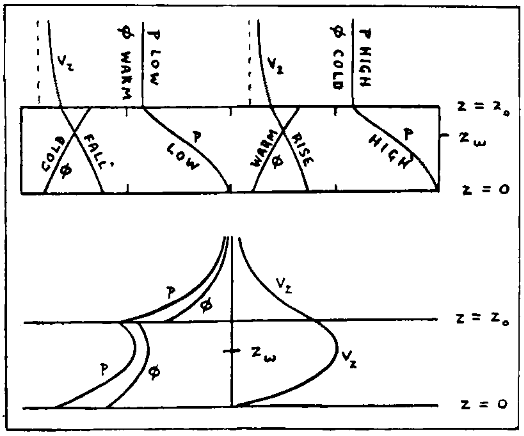

Fig. 2. Above: Phases. Below: Amplitudes.

In the above we have considered only the case of uniform thermal wind, the changes in Richardson number being due to changes in static stability. The general case, when both thermal wind and static stability change, is much less simple mathematically though it could probably be tackled by numerical methods. But from energy considerations there seems little doubt that the general behaviour of any system depends primarily on the distribution of Richardson number and much less on the way in which it is compounded.

We have still only considered systems which are horizontally of infinite extent and although strongly baroclinic regimes are often of considerable longitudinal extent (in the direction of the thermal wind), they are seldom very broad. As a further step towards realism we consider a three-regime system in which the regimes are now side by side. For simplicity we commence with the case of motion between horizontal rigid boundaries which we found previously to be a useful first approximation. We suppose the inner regime to occupy $-\frac{y_0}{2} < y < \frac{y_0}{2}$ and the outer regimes to be of indefinite extent. All three regimes occupy $-\frac{z_0}{2} < z < \frac{z_0}{2}$. Once again we suppose differences in Richardson number to be due solely to differences in static stability (consistent with a continuous temperature distribution if the stability reduction is due to cloud). We take $h = h_1$ in the inner regime, $h = h_2$ in each outer regime with $|h_1| < |h_2|$ so that the inner regime corresponds, for example, to a baroclinic cloud mass. In previous cases we found the only restriction on σ was that it must be real, though the most interesting case (maximum growth-rate) was $\sigma = 0$. In the present instance we find simple solutions only if:

$$h_1^2\,(1 + \sigma_1^2) = h_2^2(1 + \sigma_2^2) \equiv H^2 \qquad (37)$$

and then $Q = HX$ has the same interpretation everywhere. The boundary conditions at $z = \pm\frac{z_0}{2}$ are satisfied as in our first problem and then they are satisfied in all regimes simultaneously. The internal boundary conditions require continuity of p and v_y (normal velocity) at $y = \pm\frac{y_0}{2}$. (Since the winds are roughly geostrophic these conditions are nearly equivalent — hence the "fit" must be

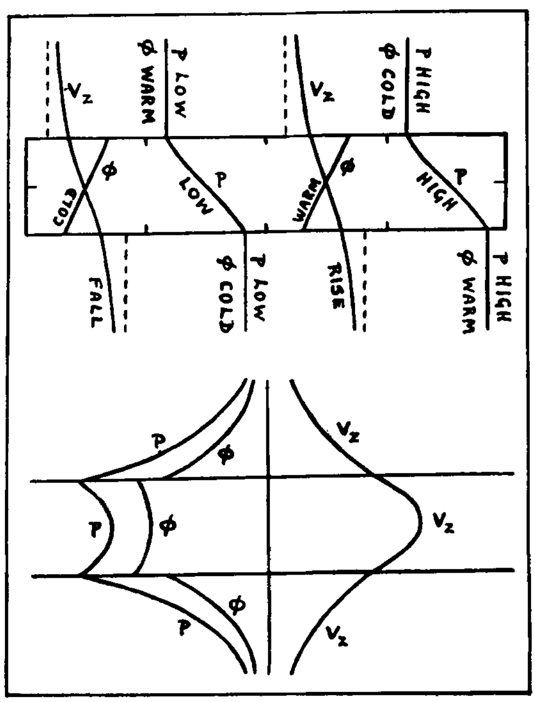

Fig. 3. Above: Phases. Below: Amplitudes.

correct to a higher order in $\frac{1}{h}\Big)$. We shall suppose (and our solutions require) that σ_1 is real, σ_2 purely imaginary, consistent with our assumption that $k_2 \equiv \frac{h_1}{h_2} < 1$. Then the boundary conditions at $y = \pm\infty$ are satisfied (all perturbations decreasing exponentially) with appropriate (opposite) choice of roots $\pm\sigma_2$ in the two outer regimes. In the inner regime we take a linear combination of the solutions corresponding to $\pm\sigma_1$ respectively and then have sufficient arbitrary constants to satisfy all the boundary conditions if, at the same time:

$$\frac{i\,\sigma_2}{\sigma_1} = \tan\left(\frac{\lambda\sigma_1}{2}y_0\right) \qquad (38)$$

or, by (II. 37)

$$\sigma_1^2\left[k_2^2 + \tan^2\left(\frac{\lambda y_0}{2}\sigma_1\right)\right] = 1 - k_2^2 \quad (39)$$

Since $k_2^2 < 1$ this equation always possesses at least one real root for $|\sigma_1|$. (If there is more than one we take the smallest — corresponding to maximum growth-rate.) For the growth-rate we have:

$$\vartheta_1 = \beta'\frac{K}{h_2}:$$

$$\beta' \equiv \frac{1}{k_2}\cdot\frac{\beta}{\sqrt{1+\sigma_1^2}}, \qquad (40)$$

where β is defined by (II. 28). Once again β' is real, and the disturbance unstable, only within the range (II. 30) and for one particular value of α growth-rate is a maximum. Since σ_1 depends on λ the "dominant" values of $|\alpha|$ and $|\beta|$ are slightly different from those given by (II. 32) — $|\alpha|$ is somewhat greater and $|\beta|$ slightly smaller by an amount depending on $\frac{y_0}{z_0}$ and on k_2. The variation of β' with α as compared with the case of an infinite cloud-sheet ($y_0 = \infty$) is shown in Fig. 4 for a typical case in which $k_2^2 = \frac{1}{10}$; $\frac{1}{\sqrt{b_2}}\frac{y_0}{z_0} = \frac{1}{4}$; (see II. 5).

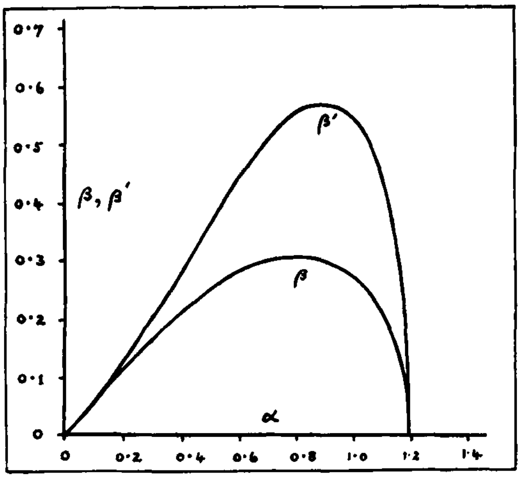

Fig. 4. Selection Curves.

It will be observed that the "selectivity" (i.e. the sharpness of the maximum) is increased. As a numerical example we may compute the characteristic features of the dominant wave of the system to which Fig. 4 applies, assuming in addition $K = 0.4$ hr^{-1} (lat. 50° approximately) : $\sqrt{b_2} = 115$ (typical for unsaturated air) : $z_0 = 5$ km : $y_0 = 150$ km (height and width of cloud mass) : $\frac{dU}{dz} = 10$ hr^{-1}. Then we derive:

$$h_2^2 = 21;\ \alpha = 0.893;\ \beta' = 0.575;\ \sigma_1 = 1.333; \qquad (41)$$

and then:

$$\frac{1}{\vartheta_1} = 19.9 \text{ hr};\ \mathbf{L} \equiv \frac{2\pi}{\lambda} = 1{,}070 \text{ Km}, \qquad (42)$$

so that the disturbance doubles its size in approximately 14 hours. The growth-rate is slightly less than that sometimes observed since in practice the effective Richardson number may be smaller than that assumed (and then our approximations are not so good — but see below). But both growth-rate and wavelength are of the right order of magnitude for cyclone waves. On the other hand when z_0 is the height of the tropopause and the Richardson number is not much less than that appropriate to unsaturated air (only cloud masses comparable in size with the disturbance pro-

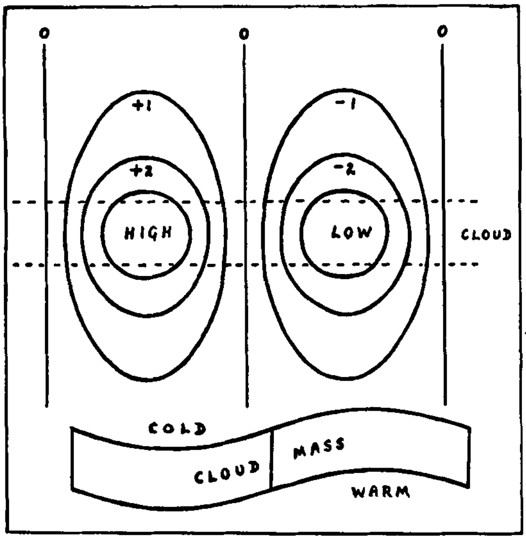

Fig. 5. Pressure-field and Cloud-mass Perturbation.

duce a large reduction in effective Richardson number) we find that **L** is more than doubled (in a typical case we find **L** is of the order of 4,000 Km) while ϑ_1 is considerably reduced $\left(\text{in a typical case } \frac{1}{\vartheta_1} = 60 \text{ hr}\right)$. These values are in good agreement with observed values in long waves. In general we find that growth-rate is intermediate between the values appropriate to infinite sheets of Richardson number h_1 and h_2 respectively, approaching the former for very broad, the latter for very narrow "cloud belts".

The structure of the "dominant" disturbance has many interesting features which will be discussed only briefly. The vertical distribution of phase and amplitude of the perturbation functions is exactly as in Fig. 1, but the horizontal structure is of course more complex. The approximate pressure-perturbation pattern $\left(\text{correct to zero order in } \frac{1}{h}\right)$ is shown in Fig. 5.

Apart from the changes of phase and amplitude indicated in Fig. 1, the pattern is the same at all levels. For comparison with observation we must superpose the unperturbed pressure field. We then find that (except when $U = 0$) closed centres are absent until the disturbance has attained a definite size (different at different levels and perhaps never attained, especially at high levels, — but this question cannot be discussed on a theory of small perturbations, which applies only to the early stages of development). Since the winds are roughly geostrophic $\left(\text{correct to zero order in } \frac{1}{h}\right)$ we may infer the approximate horizontal wind field from Fig. 5. But note that the trajectories are not even approximately along the isobars except at the steering level $z = 0$. Relative to the disturbance the air is blowing through from, say, the east at low levels and the west at high levels. The combined relative motion is associated with a deformation of the cloud mass. This deformation is shown below Fig. 5 (since it is "infinitesimal"), in correct phase for the level $z = -\frac{z_0}{2}$ (corresponding to the earth's surface). The phase varies with height and is the same as that of the entropy perturbation (warm tongue) at levels $z = 0$, $\pm \frac{z_0}{2}$, the phase-difference elsewhere being very small. Clearly the cloud-mass corresponds to the frontal region of the growing cyclone and its displacement towards cold air to the boundary of the "warm sector". It should be emphasised that we are concerned here only with the *broad* features of disturbances. Errors of detail are inevitable since actual initial systems are usually more complicated in structure than we have assumed. From this point of view it does not seem to matter very much whether or not the velocity field contains discontinuities (except when these are unusually large *and* extensive) provided the smoothed fields are the same. Moreover sharp discontinuities observed in practice are often the result of rather than the prerequisite for development. When the theoretical fields are more accurately computed (correct to the first order in $\frac{1}{h}$, the maximum attainable with present approximations) we find discontinuities of wind, pressure gradient, entropy developing along each surface of the cloud mass. This behaviour may be regarded as the realisation of a latent discontinuity — in effective static stability — between saturated and unsaturated air. These discontinuities are more complex in structure (and in some ways in better agreement with observation) than those at the plane surfaces ascribed to theoretical "fronts". Moreover the genesis of discontinuity by development is consistent with the observed "sharpening" of fronts during cyclogenesis. But a satisfactory theory of frontogenesis cannot be based on considerations of small disturbances and we must leave this aspect (from our point of view a matter of detailed structure) for the present.

It is easy to combine the virtues (from the point of view of realism) of the horizontally and vertically "stratified" systems by considering the systems shown in cross-section in Fig. 6. The only difference as compared with the system last considered, is that in place of Fig. 1, we use Figs. 2 and 3 respectively for the vertical variations of phase and amplitude provided that the conditions:

$$\frac{h_2^2}{h_1^2} = \frac{h_4^2}{h_3^2} \qquad (43)$$

are satisfied. We then find that all the boundary conditions can be satisfied simultaneously. The condition (II. 43) is necessary for solutions of simple mathematical form but it has little physical significance since very little perturbation-energy is associated with the "corner" regimes. In fact when h_1^2 is considerably smaller than the value of h^2 outside it is this alone (together with the dimensions of the cloud mass) which is the main determining factor of the features of the disturbance.

Several refinements and extensions of the theory will not be discussed in detail here. Thus the solutions of (II. 17) correct to order $\frac{1}{h^3}$ are easily determined. Applied to the first system considered they yield more precise formulae for growth-rate etc. It appears that serious errors do not result until h^2 approaches fairly close to unity. For $h^2 < 1$ a second type of instability (corresponding to "vertical overturning" — see below) becomes possible. This type of instability is associated with development on a smaller scale (motion in the vicinity of cold fronts, tropical cyclones etc.) and will be discussed elsewhere. Another extension is the calculation of second-order

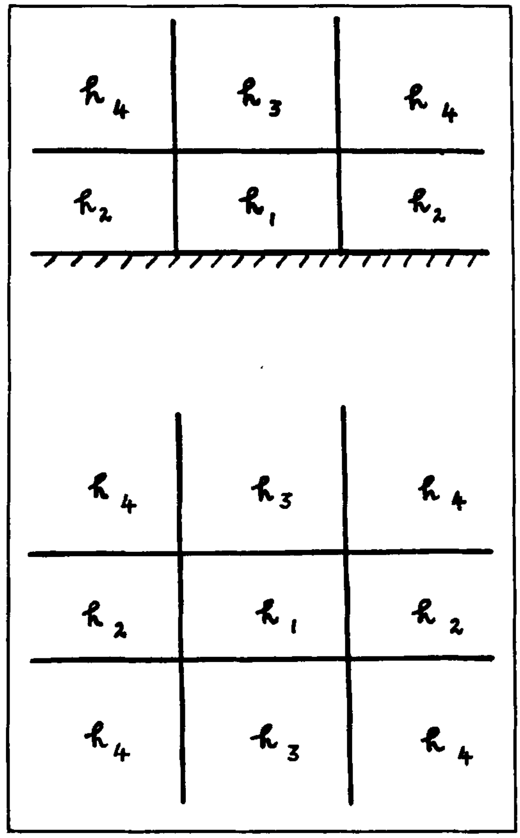

Fig. 6. 6- and 9-regime Systems (Vertical Section).

perturbations i.e. correct to the second order of small quantities. We then obtain terms involving Ψ^2 i.e., second harmonic terms. Calculations are simple for the first system, corresponding to Fig. 1. We find that the complete perturbation is no longer symmetrical as between "high" and "low". At all levels the pressure troughs are accentuated and the ridges flattened, in good agreement with observed behaviour. In fact, correct to zero order in $\frac{1}{h}$, the phase-lines of the first and second order perturbations of pressure coincide (for troughs) at all levels. We also obtain terms independent of Ψ, corresponding to "transport" phenomena. Thus for example the disturbances transport heat *upwards* and their final effect must be to *increase* the static stability of the system. (In this case of course the same result is obtained from the integrated correlation of v_z and φ.) An important corollary is that since large-scale disturbances are always transporting heat upwards we must, for statistical balance, have net cooling of the upper troposphere by radiation. Independent calculations of radiation flux have led to the same conclusion.

III. Long Wave Modifications

In the previous analysis, in which we have neglected the variability of the Coriolis parameter, we found that the disturbances move with the unperturbed current at the "steering level" which, in the case of symmetrical systems (Figs. 1 and 3) is the middle level of the system. In the system to which Fig. 2 applies the steering level is somewhat elevated, corresponding to a limited extension of the disturbance into the upper layer. If we apply the superposition (of uniform wind and corresponding geostrophic pressure gradient) theorem referred to in the first section we obtain the law of "contour steering" (*not*, as is often stated "thermal steering") at the steering level. So far at least as direction of travel is concerned this is in good agreement with observation (though we have proved the result only for nascent disturbances). But this law applies only to disturbances on not too large a scale. As we shall see, the most important new feature arising when we take into account the variation of K with latitude is a modification in the steering law.

For dealing with the more complicated problems in which we take into account the variability of the Coriolis parameter, or use a polar co-ordinate system, or are dealing with complicated boundary conditions, etc., it is convenient to reformulate the perturbation equations. This involves further approximations so that the solutions are valid only to zero order in $\frac{1}{h}$ (on the other hand in long wave problems h^2 is much larger than in cyclone problems) and the calculations are not so easily extended to the second order of small quantities but the differential equations are of a simpler form, are better adapted to numerical work and the solutions still retain the essential features we are seeking to determine. We commence with a "standard" distribution $\varrho_0(z)$, which may be the actual distribution in some central part of the region in which we are interested, and we shall suppose $p_0(z)$ the corresponding pressure distribution. Let p', ϱ' be the differences at any point from the "standard" values at the same level. Then $\frac{1}{\gamma} \cdot \frac{p'}{p_0}$, $\frac{\varrho'}{\varrho_0}$ are both small compared with unity and normally in regions of strong thermal wind the former is much smaller than the latter. Hence approximately:

$$\varphi = \frac{1}{g\varrho_0} \cdot \frac{\partial p}{\partial z} = \frac{1}{g}\frac{\partial}{\partial z}\left(\frac{p}{\varrho_0}\right) \tag{1}$$

for perturbations of sufficiently shallow systems, in which $\left|\frac{\partial}{\partial z}\log p\right| \gg \left|\frac{d}{dz}\log \varrho_0\right|$. If in the initial system the static stability and thermal wind are uniform we have for adiabatic motion (cf. II. 4):

$$a v_y + b v_z + \frac{1}{K^2} \cdot \frac{d}{dt}\frac{\partial}{\partial z}\left(\frac{p}{\varrho_0}\right) = 0 \tag{2}$$

We have noted that the perturbation winds are, to zero order in $\frac{1}{h}$, geostrophic. Then replacing v_y by $\frac{1}{K\varrho_0} \cdot \frac{\partial p}{\partial x}$ and differentiating:

$$b\frac{\partial v_z}{\partial z} + \frac{1}{K^2}\left(\frac{\partial}{\partial z}\frac{d}{dt} + Ka\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z}\left(\frac{p}{\varrho_0}\right) = 0 \tag{3}$$

neglecting once again the variation of $\log \varrho_0$. Hence by the first of equations (II. 4) and (II. 6):

$$b \cdot \mathrm{div}_H \mathbf{v} = \frac{1}{K^2} \cdot \frac{d}{dt}\frac{\partial^2}{\partial z^2}\left(\frac{p}{\varrho_0}\right) \tag{4}$$

When h^2 is large the third of equations (II. 4) approximates to:

$$-\,\mathrm{div}_H \mathbf{v} = \frac{1}{K} \cdot \frac{d}{dt}\,\mathrm{curl}_H \mathbf{v} = \frac{1}{K^2} \cdot \frac{d}{dt}\nabla_H^2\left(\frac{p}{\varrho_0}\right) \tag{5}$$

on substitution of the geostrophic winds. Combining (III. 4) and (III. 5)

$$\frac{d}{dt}\left[\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{1}{b}\frac{\partial^2 p}{\partial z^2}\right]=0 \qquad (6)$$

and this condition is certainly satisfied if:

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial(\sqrt{b}\cdot z)^2}=0 \qquad (7)$$

All the other perturbation functions are readily expressible in terms of p and using (III. 7) in place of (II. 7) we can obtain, correct only to zero order in $\frac{1}{h}$, the results of section II.

Our present approximations are therefore consistent with, albeit more drastic than those made earlier. Now if we take into account the variability of K we have, in place of (III. 5):

$$-\operatorname{div}_H \mathbf{v}=\frac{1}{K}\frac{d}{dt}(K+\operatorname{curl}_H \mathbf{v})=$$

$$=\frac{1}{K^2}\left[\frac{d}{dt}\nabla_H^2\left(\frac{p}{\varrho_0}\right)+\frac{dK}{dy}\cdot\frac{\partial}{\partial x}\left(\frac{p}{\varrho_0}\right)\right] \qquad (8)$$

and then, in place of (II. 6)

$$\frac{d}{dt}\left\{\nabla_H^2 p+\frac{1}{b}\frac{\partial^2 p}{\partial z^2}\right\}+\frac{dK}{dy}\cdot\frac{\partial p}{\partial x}=0 \qquad (9)$$

For solutions of the type (II. 8 and II. 13) studied in the previous section (III. 9) leads to the ordinary differential equation:

$$\frac{d^2G}{dP^2}=\left(\frac{1}{C^2}-\frac{1}{P}\right)G \qquad (10)$$

where

$$C^2\equiv\frac{h^2}{(\lambda^2+\mu^2)}\cdot\left(\frac{1}{K}\frac{dK}{dy}\right)^2$$

$$P\equiv CQ=Ch\sqrt{1+\sigma^2}\cdot X. \qquad (11)$$

From (III. 2) and (III. 10) we obtain the relation:

$$N=-P^2\frac{d}{dP}\left(\frac{G}{P}\right) \qquad (12)$$

and in place of (III. 10) we may use:

$$Q(Q-C)\frac{d^2N}{dQ^2}-(2Q-C)\frac{dN}{dQ}-$$

$$-(Q-C)^2N=0 \qquad (13)$$

which of course reduces to (II. 23) when $C = 0$. (To our present approximation N and R are equivalent.) The general solutions of (III. 10) and (III. 13) are expressible in terms of Whittaker functions but since we are interested in functions of a complex variable evaluation of the solutions and determination of the dominant solution is in general a laborious process. If however, in order to discover the initial effect of the term involving $\frac{dK}{dy}$ when the "correction" is not too large, i.e. for waves which are not too long, we assume $|C| \ll 1$ then (III. 13) approximates to:

$$Q\frac{d^2N}{dQ^2}-2\frac{dN}{dQ}-QN=$$

$$=C\left[\frac{d^2N_0}{dQ^2}-\frac{1}{Q}\frac{dN_0}{dQ}-2N_0\right] \qquad (14)$$

(where N_0 corresponds to $C = 0$), which is easily solved by variation of parameters (the L. H. S. is the same as in II. 23), the solutions involving exponential integrals. If we consider now the first initial system (two rigid boundaries) and compare our solutions for C small but non-vanishing with our original solutions for $C = 0$ we find that the dominant wavelength and the corresponding value of ϑ_1 are unaltered, but that we obtain an additional term in the real part of the wave-velocity, corresponding to a lowering of the steering-level, given by:

$$-\delta U_\omega=0.726\cdot\frac{1}{(\lambda^2+\mu^2)}\cdot\frac{dK}{dy} \qquad (15)$$

a formula differing only by a numerical factor from that applicable to (hypothetical) barotropic waves. The formula appears to be in reasonably good agreement with observation in middle-high latitudes (poleward of 45° lat.) where the *dominant* wavelength is typically about 4,000 Km (though we have proved the result only for *growing* waves).

The steering-level corresponds roughly with minimum amplitude of the pressure perturbation (cf. Fig. 2) and the present lowering of the steering-level is associated with an increase of perturbation amplitude at high levels (i.e., towards the tropopause) and a decrease at low levels. That long waves are more intense at high as compared with low levels is of course well known.

We have observed that the decrease of mean density with height involves a "distortion" which is most serious in the case of deep waves such as long waves. An approximation to the modifications involved is obtained by replacing the first of equations (II. 4) by:

$$\mathrm{div}_H \mathbf{v} + \frac{\partial v_z}{\partial z} - \frac{1}{z_c} \cdot v_z = 0 \qquad (16)$$

where

$$\frac{1}{z_c} = -\frac{d}{dz} \log \varrho_0 = \frac{g}{\gamma R T} \qquad (17)$$

Then in place of (II. 23) we obtain finally:

$$\left[Q \frac{d^2R}{dQ^2} - 2\frac{dR}{dQ} - QR\right] + m\left[Q\frac{dR}{dQ} - 2R\right] = 0 \qquad (18)$$

where

$$m \equiv \frac{1}{Ha\lambda z_c} = -\frac{1}{2\alpha} \cdot \frac{z_0}{z_c} \qquad (19)$$

(cf. II. 29, II. 5 and II. 2). Once again we shall consider the initial form of the "correction" when it is small, i.e. when z_0 is appreciably less than $1.6\ z_c$. Making the substitution:

$$R = e^{-\frac{m}{2}Q} \cdot R' \qquad (20)$$

and neglecting m^2 we get in place of (III. 18):

$$Q\frac{d^2R'}{dQ^2} - 2\frac{dR'}{dQ} - (Q + m)\, R' = 0 \qquad (21)$$

Using once again the method of variation of parameters we obtain the general solution of (III. 21) and can study the modifications resulting from small but non-vanishing m as compared with our original solutions when $m = 0$. As above, when we were concerned with the effect of $\frac{dK}{dy}$, we find that the formulae for dominant wavelength and growth-rate are unaffected but there is an additional term in the real part of the wave-velocity. Expressed in terms of the steering-level we find that the latter is depressed by an amount $-\delta z$ where:

$$-\frac{\delta z}{z_0} = 0.0875 \left(\frac{z_0}{z_c}\right) \qquad (22)$$

Even when, as is the case in practice, $\frac{z_0}{z_c}$ is comparable with unity, and the correction is rather rough, this depression of the steering-level due to decrease of density with height is much smaller, for long waves, than that due to variation of the Coriolis parameter. To our present approximations these effects are of course additive. We may note that the factor $e^{-\frac{m}{2}Q}$ in (III. 20) is simply $\frac{1}{\sqrt{\varrho_0}}$ multiplied by a (complex) constant. Thus apart from the difference in behaviour of the amplitudes of perturbations which are functions of R' as compared with those of the R of (II. 23), a relatively minor difference, we find that perturbation amplitudes are multiplied by $\frac{1}{\sqrt{\varrho_0}}$, corresponding to increase in relative amplitude with height (and, to this extent, constancy of wave-energy density) and further accentuating the feature, already noted above, that long wave amplitude increases (on the whole) with height. Of course the corrections (III. 15) and (III. 22) apply to cyclone waves as well as long waves. But since both corrections are proportional to the square of the wavelength they can usually be neglected for practical purposes in the former case.

For practical application we have identified z_0 for long waves as the height of the tropopause since the mean value of h^2 over a considerable depth of the stratosphere is normally considerably larger, in significant regions, than in the troposphere. For a closer approximation we may study a system, similar to that to which Fig. 2 applies, in which the disturbance extends (but with exponential decrease with height) into the stratosphere. We cannot immediately apply our previous results since the sign of $\frac{dU}{dz}$ in the upper regime is reversed and the internal boundary (i.e., the tropopause) is no longer horizontal so that a precise mathematical formulation leads to a more complicated problem. Nevertheless it is easy to see that the two cases are not very different (and incidentally provide confirmatory evidence that the essential characteristics of a system depend on the Richardson number rather than its component elements). For the solutions R_1, R_2 (equation II. 24) are simply interchanged (h replaced by $-h$) when $\frac{dU}{dz}$ is reversed. In order to satisfy the boundary conditions at $z = \infty$ we have to choose the same solution (i.e., that which decreases with height) as before. We shall suppose the change in h^2 at the tropopause large (k_1 small) so that decrease in the stratosphere is rapid and most of the wave-energy in this regime concentrated just above the tropopause. The distribution of $\text{curl}_H \mathbf{v}$ in the lower regime, and in particular at the tropopause, cannot differ much from the limiting case (Fig. 1 — cf. Fig. 2). Since there are no discontinuities of velocity in our initial system we find that $\text{curl}_H \mathbf{v}$ must be continuous (at least to zero order in $\frac{1}{h}$, the winds being roughly geostrophic) at the tropopause. Hence $\text{curl}_H \mathbf{v}$ in the lower stratosphere is not much altered when $\frac{dU}{dz}$ is reversed. And just above the tropopause U is only slightly reduced by this reversal. Hence $\frac{\partial v_z}{\partial z}$, which is determined to this order of accuracy, by $\left(U\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$ $\text{curl}_H \mathbf{v}$ is not much altered and this is consistent with our choice of R_1 or R_2 corresponding to v_z decreasing. In Fig. 2 both v_y and v_z are continuous at the internal boundary, the resultant velocity having a slope smaller than that of the isentropic surfaces in the lower regime but greater than that in the upper regime and producing the 180° phase change in φ. When $\frac{dU}{dz}$ in the stratosphere is reversed the slope of the isentropic surfaces there is reversed but there is no qualitative (and only a relatively small quantitative) change in the above description. In fact in the significant region (i.e., just above the tropopause) we can reverse the sign of $\frac{dU}{dz}$ and our original solution (Fig. 2) is still a rough approximation to the required solution. Alternatively we may imagine the stratospheric value of $\frac{dU}{dz}$ changing gradually through zero. The general form of the solution in the stratosphere does not change appreciably, only the scale, which depends on h^2, *not* on the sign of $\frac{dU}{dz}$. We may therefore still use Fig. 2 as an approximate description in the more realistic case. In particular we find that the effect of this refinement is a raising of the steering-level. It is a convenient accident that in the case of typical long waves this correction very roughly cancels the correction for variation of ϱ_0 (depression of steering-level) so that we may obtain quite accurate steering velocities by omitting both corrections.

When discussing the system to which Fig. 2 applies we noted the sucking down of the internal boundary in the vicinity of the upper trough (or low pressure centre) and its pushing up in the vicinity of the upper ridge. We may now apply this result directly to the tropopause when there is long wave development. Apart from the agreement with observation so far as the tropopause itself is concerned we may note that we have here a mechanism ("advection" plus "stretching") to account quantitatively for observed changes in ozone measurements by purely dynamical considerations.

IV. Energy Analysis

The above account is based on obtaining complete solutions of the perturbation equations for certain simple initial systems thereby proving the systems unstable and determining the manner of breakdown. It is instructive to consider an alternative analysis in which we merely show the *possibility* of instability.

Nevertheless this more limited analysis makes clear the general nature of the process and makes possible an estimate of maximum growth rate for disturbances (strictly it determines an absolute upper limit to growth-rate) of a wider range of initial systems.

We commence with a system (such as one of those with which we have been concerned) initially in equilibrium and imagine displacements δx, δy, δz at each point, in general functions of x, y, z, t. For simplicity we consider the "incompressible" case (or analogue). The loss in potential energy is computed correct to the second order of small quantities and equated to the gain in kinetic energy, i.e., the kinetic energy associated with a growing disturbance. If we confine our attention to disturbances of "constant shape" in which all the displacements contain the factor $e^{\vartheta_1 t}$ then the kinetic energy involves terms such as $\vartheta_1^2 \cdot \delta y^2$ etc. In this way we obtain ϑ_1^2 as a function of the displacements and, making use of the constraints implied by the continuity equation and one of the momentum equations, we determine an upper limit to ϑ_1^2. If this upper limit is positive we infer that the system is unstable "potentially", i.e., subject to compatibility with other constraints not considered. (In all the cases examined we are able to find physically possible boundary conditions such that this upper limiting growth-rate is actually attained within an arbitrarily small amount. Other boundary conditions yield smaller, but in general comparable, maximum growth-rates. It appears to be a general rule that "potentially" unstable systems are actually so, a feature which may be associated with the infinitude of degrees of freedom of fluid motion.) Apart from the change in sign of ϑ_1^2 the method is essentially the same as that used by Rayleigh to determine minimum frequencies of oscillation.

Applied to a barotropic system with uniform velocity this analysis gives an upper limit to the initial rate of growth of a Bénard cell:

$$\vartheta_1^2 \leq -gB \qquad (1)$$

Growth-rate is a maximum when δz is everywhere large compared with δx, δy and there is instability only when $B < 0$.

Now consider a baroclinic system. Potential energy is released by a process of "overturning" (we consider the initial stage of this process). It will be convenient to consider separately "vertical overturning" in a plane at right angles to the thermal wind and "quasi-horizontal overturning" in a sloping plane parallel to the thermal wind. The latter case corresponds to the disturbances with which we have hitherto been concerned and it will be convenient to take this case first. Assuming that the disturbances are periodic in the direction of the thermal wind we obtain finally the result:

$$\vartheta_1^2 \leq -g_s \frac{\partial \Phi}{\partial s} \qquad (2)$$

if overturning takes place in the xs plane (the s axis being in the yz plane) and g_s is the component of gravity along the s-axis. By hypothesis the isentropic surfaces are not horizontal and if α is their angle of slope (acute angle) we find that ϑ_1^2 is positive if the s-axis slopes at a smaller angle. If α is small we find that growth-rate is a maximum when the s-axis has a slope $\frac{\alpha}{2}$ and then:

$$\vartheta_1^2 \leq \frac{1}{4} g \frac{A^2}{B} \qquad (3)$$

and from the definition (II. 12) this is the same as:

$$|\vartheta_1| \leq \frac{1}{2} \cdot \frac{K}{h} \qquad (4)$$

Our complete solution (II. 31) therefore corresponds to about 62 % efficiency. The reduction is of course due to the constraints of the rigid boundaries which prevent all the fluid particles being displaced in the optimum direction. (But it is easily verified that in the central region, near $z = 0$, the displacements are nearly in the optimum direction, i.e., along the bisector of the angle between the isentropic surfaces and the horizontal.)

The analogy between (IV. 2) and (IV. 1) is evident — we are here concerned with a kind of "convection" on a large scale, the main displacements being not vertical but in the direction of the s-axis. (Isentropic charts sometimes suggest this kind of picture.)

In the above we have supposed U independent of y as in the systems for which we have obtained complete solutions. We may attempt to generalise the above analysis by considering a system in which initially $V_x = U(z) + W(y)$ but for solutions periodic in the x-direction we have to abandon the assumption of "constant shape". (We are confronted with difficulties similar to those arising in a study of the stability of Couette flow). On the other hand it is easy to study the effect of vertical overturning (in the yz plane) in such a system. We obtain finally:

$$2\,\vartheta_1^2 \leq -\left[gB + K\left(K - \frac{dW}{dy}\right)\right] + \sqrt{\left[gB+K\left(K-\frac{dW}{dy}\right)\right]^2 + 4\left[(gA)^2 - gB\cdot KK - \left(\frac{dW}{dy}\right)\right]} \quad (5)$$

and if we suppose (as is usually the case):

$$B > 0 : \frac{dW}{dy} < K\,; \quad (6)$$

the condition for "potential" instability is:

$$(gA)^2 > gB \cdot K\left(K - \frac{dW}{dy}\right) \quad (7)$$

which is the same as:

$$\frac{1}{h^2} > \left(1 - \frac{1}{K}\frac{dW}{dy}\right) \quad (8)$$

There is instability for any Richardson number if $\frac{dW}{dy} > K$, i.e., for an anticyclonic wind shear greater than K, a well-known result but probably not one of very great importance from a practical point of view. On the other hand when $\frac{dW}{dy} \ll K$, (IV. 8) becomes approximately $h^2 < 1$. Thus whereas the atmosphere is normally unstable from the point of view of "quasi-horizontal overturning" it is only so in special circumstances from the point of view of "vertical overturning". We should therefore expect the former process to be dominant in atmospheric development and observation appears to confirm this result at least in middle and high latitudes. (In low latitudes any kind of development is slow unless the static stability is small or negative.) Now the overturning process is an irreversible one and "quasi-horizontal overturning" leads to interchange between warm air at low levels and cold air at high levels. As noted earlier heat is transported *upwards*, statistically balancing radiative cooling at high levels. At the same time heat is normally transported polewards (the main transport is probably associated with long waves) to balance statistically net radiative cooling in high latitudes.

V. The Ultimate Limitations of Weather Forecasting

We may infer from the above analysis (in so far as the atmosphere is always, on a large scale, baroclinic) that atmospheric motion is normally unstable. The fact that practical systems are usually more complicated than those studied does not affect the generality of this result. In fact we have in practice what may legitimately be described as "fully developed turbulence" of a particular kind, the turbulent motion being maintained against frictional dissipation by the growth, from time to time, of disturbances of the kind we have been studying. (The only essential difference between this large-scale turbulence and that occurring on a smaller scale is the manner in which energy is supplied to the turbulent disturbances). Assuming sufficient analytical skill, what are the possibilities of forecasting for such a system? Suppose we attempt to formulate the problem as one of determining a final (forecast) state from a given initial one. The initial state is in practice "given" only within a certain margin of error. For concreteness let us consider pressure at a given point, known within a margin δp. Let ϑ_1 be the maximum growth-rate of unstable disturbances of the system. Then in the final (forecast) state we can guarantee pressure correct only within a margin $\delta p \cdot e^{\vartheta_1 t}$ since disturbances below the margin of error initially (and therefore completely unknown) will have attained this size. It is clear that "guaranteed" forecasts are possible (even in theory) only for intervals less than t_1 where t_1 is of the order of $\frac{1}{\vartheta_1}$, for beyond this time the margin of uncertainty is so large as to make such "guaranteed" information valueless. Reduction of initial error-margin makes possible only a very limited extension of time interval. For larger time intervals we must reformulate our problem.

Although we cannot, with complete certainty, say anything about long-term develop ments $\left(t \gg \frac{1}{\vartheta_1}\right)$ it does not follow that all possible developments are equally probable. On the contrary we may infer that probabilities are very unequally distributed (and therefore that information of this kind may be, from a practical point of view, almost as good as "guaranteed" information). As an example consider a set of unstable disturbances of various growth-rates. So long as the determining (perturbation) equations are substantially linear it is clear that the relative importance of the disturbance of maximum growth-rate increases with time i.e., any disturbance composed of components of varying growth-rates will tend towards the size, structure and growth-rate of the "dominant" wave by a process of "natural selection". We may generalise this result by including stable components in the initial perturbation and there can be little doubt that the result is true of *almost* any arbitrary disturbance. Of course in practice conditions are more complicated, the concept of an initial system is less clear-cut and the dominant disturbance is a relatively slowly varying function of the time, quite apart from the modifications which ensue when the disturbance becomes "finite" and the governing equations non-linear. (Moreover we may *choose* to regard certain initial irregularities as "finite" perturbations of a larger system.) Nevertheless the reality of the selection process is made clear every day when we see recognisable, well-known patterns developing "as if from nowhere" which more or less closely resemble in size, structure and behaviour the ideal disturbances we have discussed theoretically.

The above is no more than a prelude to the rather formidable task facing theoretical meteorology — that of discovering the nature of and determining quantitavely all the forecastable regularities of a "permanently unstable" (i.e., permanently turbulent) system. We can be certain that these regularities are necessarily statistical and to this extent our technique must resemble statistical mechanics. But we do not yet know enough about the "atoms" (the life-histories of disturbances) nor are we concerned with "atoms" with a clear-cut individuality. Clearly there are difficulties from the point of view of formulation and it is by no means clear what kind of problem we ought to attempt to solve. But these difficulties are inherent in the study of any kind of turbulent motion and perhaps in the study of irreversible processes (other than isolated ones) in general.

Much of the above formed the subject of a series of colloquia given, at the kind invitation of Profs. J. Bjerknes and C. L. Godske, at the Geofysisk Institutt, Bergen in April, 1947. A more detailed treatment of cyclone theory was given in a doctoral thesis (unpublished: London, 1948). The literature on cyclone and long wave theory is extensive and the writer would like to be excused the compilation of a list of references. He would however like to refer to an independent analysis by J. G. Charney (Journal of Meteorology, Vol. 4, No. 5, Oct., 1947) which in many (but not all) respects is consistent with his own.