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A CONTRIBUTION TO THE PROBLEM OF DEVELOPMENT

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SUMMARY

Subject to various approximations it is shown that the relative (isobaric) divergence between two pressure levels is given by

$$l\,(\mathrm{div}_p\,\mathbf{V} - \mathrm{div}_p\,\mathbf{V}_o) = -V'\frac{\partial}{\partial s}(l + \zeta + \zeta_o)$$

$$= -V'\frac{\partial}{\partial s}(l + \zeta' + 2\zeta_o)$$

where V' is the relative wind (or shear), l the Coriolis parameter, ζ and ζ_o the vorticities at the two levels and $\frac{\partial}{\partial s}$ denotes differentiation in the direction of shear

The formula can be used to explain thermal steering and other development processes and is adaptable to the routine forecasting problem.

1. In an earlier paper (1939) it was argued that development in the atmosphere consists of a vertical distribution of divergence and convergence such that the integrated divergence, defining the surface pressure tendency, is a small residual of opposing contributions at different levels. Typical cyclogenesis consists of convergence in the lower troposphere approximately balanced by divergence above probably mainly in the upper troposphere. Typical anticyclogenesis has the reverse structure. It was inferred that development could be diagnosed by taking the difference between the divergence at the surface and some upper level which would normally be in the upper limb of the circulation. This method would have the simplifying effect of cancelling out the contribution of the surface tendencies which are reflected (hydrostatically) at all levels and so avoiding the difficulty that in relating development to the surface isallobaric field the question is being essentially begged.

It was further argued, ignoring the effect of changing latitude that the geostrophic wind field is non-divergent which (in a frictionless atmosphere) means that the divergence is defined by that of the geostrophic departure or by the curl of the horizontal acceleration. Thus a study of the field of acceleration difference between the surface and upper level would indicate the nature of development. The present paper develops the same general argument in terms of the change of vorticity which, closely connected with the curl of acceleration, has received considerable attention by Rossby (1939, 1940) and others.

2. It will be convenient to employ, where applicable, the isobaric co-ordinates used by Sutcliffe and Godart in a Meteorological Office Memorandum (1942) In this system the pressure p is taken as the vertical variable and the behaviour of the atmosphere is

represented by functions of the independent variables x, y, p, t Vertical height above sea-level, z is now a dependent variable. Partial differentials in this system are distinguished by the suffix p.

As previously shown the transformations' are

$$\left(\frac{\partial}{\partial x}\right)_p = \frac{\partial}{\partial x} - \frac{\partial p}{\partial x} \cdot \frac{\partial}{\partial p}$$

$$\left(\frac{\partial}{\partial y}\right)_p = \frac{\partial}{\partial y} - \frac{\partial p}{\partial y} \cdot \frac{\partial}{\partial p}$$

$$\frac{\partial}{\partial p} = - \frac{1}{gp} \frac{\partial}{\partial z}$$

$$\left(\frac{\partial}{\partial t}\right)_p = \frac{\partial}{\partial t} - \frac{\partial p}{\partial t} \cdot \frac{\partial}{\partial p}$$

we also have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial}{dt}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \frac{dp}{dt} \cdot \frac{\partial}{\partial p}$$

With a hydrostatic field of pressure the equation of continuity takes the simple form (due to Godart)

$$\left(\frac{\partial u}{\partial x}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p + \frac{\partial}{\partial p} \cdot \frac{dp}{dt} = 0$$

or

$$\frac{\partial}{\partial p} \cdot \frac{dp}{dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p \equiv \text{div}_p \mathbf{V} \quad . \quad . \quad (1)$$

Integrating

$$\frac{dp}{dt} = -\int_0^p \text{div}_p \mathbf{V} \cdot dp \quad . \quad . \quad (2)$$

Here dp/dt, the highly significant thermodynamical quantity defining dynamical development and adiabatic processes in the atmosphere, is seen to be directly defined by the integral of the isobaric divergence of velocity above.

3. At the earth's surface

$$\frac{dp_0}{dt} = -\int_0^{p_0} \text{div}_p \mathbf{V} \, dp \quad . \quad . \quad (3)$$

The argument that the integrated divergence* is a small residual of opposing effects at different levels is seen therefore to reduce to the generally accepted principle that the thermo-dynamical effect of surface pressure changes is small compared with the effect of vertical motion which provides values of dp/dt in the free atmosphere of much greater magnitude.

4. If dp_0/dt were taken as strictly zero we might, at least theoretically, plot the divergence as a function of pressure in a vertical

* "Isobaric divergence," is, of course, not identical with horizontal divergence, but the difference, which for most purposes is negligible, does not concern our present argument.

column through the atmosphere as by the full lines in Fig. 1. The total area between this curve and the ordinate representing zero divergence is then zero. If we construct, on an appropriate scale the integral curves (shown by the broken lines) we have a representation of the development index dp/dt at all levels.

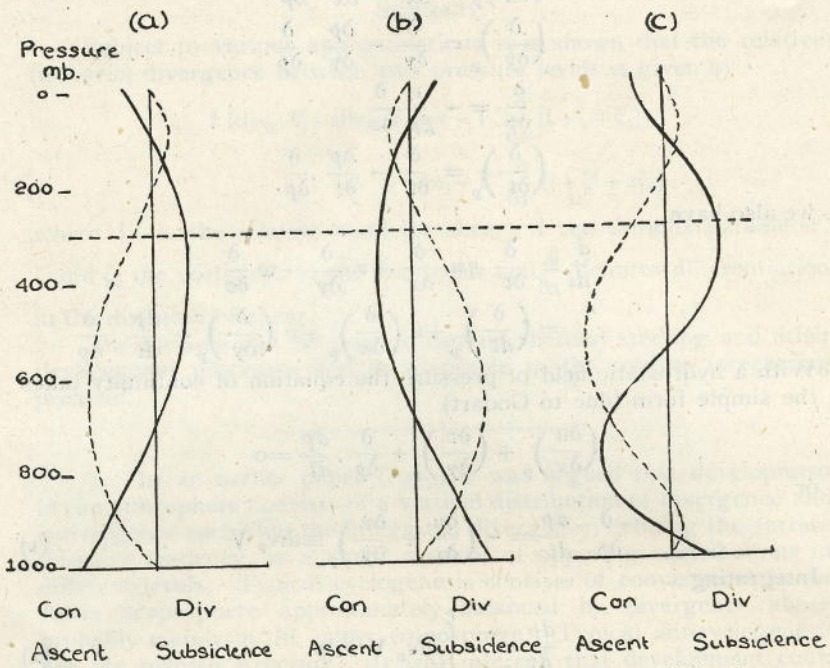

FIG. 1.—Schematic representation of the distribution with height of the isobaric divergence (full lines) and of dp/dt (broken lines).
(a) Simple cyclogenetic type.
(b) Simple anticyclogenetic type.
(c) A possible more complex type.

The theory of the 1939 paper, supported by the results to be given in this paper, suggests that the relative divergence (referred to that at the surface) will, in certain simple cases, tend to increase (or decrease) progressively with the increasing shear or thermal wind through the troposphere and then decrease (or increase) with the normally reversed thermal wind in the stratosphere. Thus Fig. 1 (a) and 1 (b) may represent distributions typical respectively of simple cyclonic and anticyclonic types of development. If we define "subsidence" and "ascent" by the thermodynamically significant index dp/dt being positive and negative respectively it is of interest to see in this graphic way how "subsidence" or "ascent" may extend through the whole troposphere into the stratosphere although the divergence changes sign at some intermediate level. It is also illuminating to speculate on possible more complex structures. Fig. 1 (c) for example represents a situation in which there is low level divergence and "subsidence" but with "ascent" through a deep layer of the atmosphere above. Clearly in such a case no indication of the nature of the significant development is to be obtained by a study only of the dynamics of the surface motion.

This digression may be of value as a warning to expect the appearance of abnormal weather developments in complex thermal structures without indication from the surface observations (including pressure tendencies).

The main point which the diagrams are intended to illustrate, is, however, that, with dp_0/dt negligibly small, the distribution of divergence and hence of dp/dt with height can be inferred from a knowledge of the relative divergence referred to some standard level (such as the surface) for then the shape of the divergence curve is known and the zero ordinate can be inserted by balancing the areas. While the surface divergence itself cannot be inferred directly without a pre-knowledge of the surface pressure changes the variation of divergence with height will be shown to be in some degree more tractable.

5. We now introduce the dynamical aspect with the equations of motion :—

$$\frac{du}{dt} = -g\left(\frac{\partial z}{\partial x}\right)_p + lv = -\left(\frac{\partial h}{\partial x}\right)_p + lv$$

$$\frac{dv}{dt} = -g\left(\frac{\partial z}{\partial y}\right)_p - lu = -\left(\frac{\partial h}{\partial y}\right)_p - lu$$

where h is the geopotential.

These give by differentiation and addition

$$\left(\frac{\partial}{\partial x}\cdot\frac{dv}{dt}\right)_p - \left(\frac{\partial}{\partial y}\frac{du}{dt}\right)_p = -\left(\frac{\partial}{\partial x}lu + \frac{\partial}{\partial y}lv\right)_d$$

$$\text{or } \mathrm{curl}_p\,\dot{\mathbf{V}} = -l\,\mathrm{div}_p\,\mathbf{V} - \frac{dl}{dt} \quad . \quad (4)$$

which is essentially the Bjerknes' Circulation theorem for the case of an isobaric element (where the solenoidal term vanishes).

It is noted then that divergence which defines the development index dp/dt by equation (2) also determines the curl of the acceleration or the rate of increase of circulation round an elementary isobaric element.

6. Equation (4) is of fundamental importance as showing how circulation is generated by divergence but unfortunately, in the practical problem, the curl of the acceleration is hardly more easy to determine than the divergence itself. We therefore transform into terms of the "isobaric vorticity":

$$\mathrm{curl}_p\,\mathbf{V} \equiv \left(\frac{\partial v}{\partial x}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p \equiv \zeta$$

we have

$$\mathrm{curl}_p\,\dot{\mathbf{V}} = \frac{d}{dt}\mathrm{curl}_p\mathbf{V} + \mathrm{curl}_p\mathbf{V}\,.\,\mathrm{div}_p\mathbf{V} + \frac{\partial v}{\partial p}\left(\frac{\partial}{\partial x}\frac{dp}{dt}\right)_p - \frac{\partial u}{\partial p}\left(\frac{\partial}{\partial y}\frac{dp}{dt}\right)_p$$

whence, by equation (4)

$$\frac{d}{dt}(\zeta + l) = -(\zeta + l)\,\mathrm{div}_p\mathbf{V} - \frac{\partial v}{\partial p}\left(\frac{\partial}{\partial x}\cdot\frac{dp}{dt}\right)_p + \frac{\partial u}{\partial p}\left(\frac{\partial}{\partial y}\cdot\frac{dp}{dt}\right)_p \quad (5)$$

or

$$\frac{d}{dt}(\zeta + l) = (\zeta + l)\frac{\partial}{\partial p}\cdot\frac{dp}{dt} - \frac{\partial v}{\partial p}\left(\frac{\partial}{\partial x}\frac{dp}{dt}\right)_p + \frac{\partial u}{\partial p}\left(\frac{\partial}{\partial y}\frac{dp}{dt}\right)_p \quad (6)$$

$(\zeta+l)$ may be called the total (cyclonic) vorticity, the sum of the apparent vorticity ζ (relative to the earth) and l due to the rotation of the earth and the expression shows that the rate of change of this quantity is related with the field of the development index dp/dt but not in any simple manner.

We may revert for a moment (in view of the greater familiarity) to the use of z as a vertical variable with $\frac{\partial}{\partial p} = -\frac{1}{gp}\frac{\partial}{\partial z}$. Since $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial x}$ are altogether negligible compared with $\frac{\partial v}{\partial z}$ and $\frac{\partial u}{\partial z}$ we may regard $-\frac{\partial v}{\partial z}$ and $+\frac{\partial u}{\partial z}$ as the components of vorticity about the axes of x and y, say ξ, η, thus

$$-\frac{\partial v}{\partial z} = \xi\,;\ \frac{\partial u}{\partial z} = \eta$$

and equation (6) may be written

$$\frac{d}{dt}(\zeta+l) = l\frac{\partial}{\partial p}\cdot\frac{dp}{dt} - \frac{1}{gp}\left[\xi\left(\frac{\partial}{\partial x}\right)_p + \eta\left(\frac{\partial}{\partial y}\right)_p + \zeta\frac{\partial}{\partial z}\right]\frac{dp}{dt} \quad (7)$$

The last term involves the product of the three-dimensional vorticity and the variation of $\frac{dp}{dt}$ in the direction of the vortex line and is not an easy quantity to deal with either theoretically or practically.*

We may remark however that the three-dimensional vorticity vector is large in regions where the vertical shear $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$ is large, that is in regions of frontal type with large horizontal temperature gradients and observation shows that the up- or down-sliding motions tend to be concentrated in such regions and to be parallel with the vortex lines. In these situations therefore the total quantity

$$\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)\frac{dp}{dt}$$

is generally small compared with the separate parts

$$\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)\frac{dp}{dt} \text{ and } \zeta\frac{\partial}{\partial z}\frac{dp}{dt}$$

Now ζ, except in the immediate vicinity of frontal regions or the centres of intense depressions, is (in extra-tropical latitudes) small compared with l. Thus it is reasonable to assume that in large-scale synoptic developments the last term in equation (7) is always small compared with the second.

We have therefore some justification for proceeding on the assumption that equation (5) may be approximated by

$$\frac{d}{dt}(\zeta+l) = -l\,\mathrm{div}_p\mathbf{V} \quad . \quad . \quad . \quad . \quad (8)$$

* This is the expression for $d\zeta/dt$ on a non-rotating earth. Compare Lamb's *Hydrodynamics*, p. 198.

In the left hand side of this expression there is however another term involving the development index for

$$\frac{d}{dt}(\zeta+l)=\left[\left(\frac{\partial}{\partial t}\right)_p+u\left(\frac{\partial}{\partial x}\right)_p+v\left(\frac{\partial}{\partial y}\right)_p+\frac{dp}{dt}\frac{\partial}{\partial p}\right](\zeta+l)$$

$$=\left[\left(\frac{\partial}{\partial t}\right)_p+\mathbf{V}\cdot\nabla_p\right](\zeta+l)+\frac{dp}{dt}\cdot\frac{\partial\zeta}{\partial p}$$

This last quantity may be compared with $\zeta\frac{\partial}{\partial p}\cdot\frac{dp}{dt}$ already neglected above. On general considerations the quantities $\frac{1}{dp/dt}\cdot\frac{\partial}{\partial p}\frac{dp}{dt}$ and $\frac{1}{\zeta}\cdot\frac{\partial}{\partial p}\zeta$ are likely to have similar magnitudes for dp/dt and ζ are likely to vary in sign or in general magnitude over similar vertical ranges of pressure. There is thus no justification for retaining one term and not the other and we shall therefore use equation (8) in the form

$$\left[\left(\frac{\partial}{\partial t}\right)_p+\mathbf{V}\cdot\nabla_p\right](\zeta+l)=-l\,\text{div}_p\mathbf{V} \quad . \quad . \qquad (9)$$

In the absence of divergence the total vorticity, taken in the isobaric surface, changes by simple advection.*

The pressure field

7. The derivation of the circulation equation, it should be noted, is obtained by eliminating the pressure (or geopotential) field from the equations of motion. Since the change in the pressure field is an essential feature of the problem it is unlikely that any result of practical value can be obtained on this basis alone. As in all problems of hydrodynamics a general solution to fit all types of motion is not practicable and it is necessary to define the type of motion before much progress is possible. Thus, for example, the cases of zero acceleration, steady states or periodic motions of special type may be amenable to solution. In nature however such simple situations rarely arise and if they do they have little practical interest. There is thus no obvious way in which the problem can be attacked with any generality.

We shall therefore be content to deal here with development only in so far as the motion is quasi-geostrophic, that is to say such that the wind velocity remains always in approximate balance with the pressure gradient. If $\mathbf{V}_g$ is the geostrophic velocity and $\mathbf{V}_{ag}$ the ageostrophic velocity we have

$$\mathbf{V}=\mathbf{V}_g+\mathbf{V}_{ag}$$

$$\text{curl }\mathbf{V}=\text{curl }\mathbf{V}_g+\text{curl }\mathbf{V}_{ag}$$

$$\frac{d}{dt}\text{curl }\mathbf{V}=\frac{d}{dt}\text{curl }\mathbf{V}_g+\frac{d}{dt}\text{curl }\mathbf{V}_{ag}$$

* The equivalent of equation (9) appears in the literature in various approximate forms, most frequently as $\frac{d}{dt}(\zeta+l)=-(\zeta+l)div\mathbf{V}$ obtained by treating the motion as horizontal. It is thought that this form is however not a consistent approximation and that (9) above is to be preferred.

and we shall assume that neglecting $\mathbf{V}_{ag}$ and its derivatives in each of the equations provides a valid first approximation. This may be difficult to justify analytically or observationally, at least for the derivatives, but there is no reason to suppose that the ageostrophic components become more important in the derivatives than in the actual wind in the case of vorticity. It is of course not valid to ignore the divergence of $\mathbf{V}_{ag}$ in

$$\operatorname{div}\mathbf{V} = \operatorname{div}\mathbf{V}_g + \operatorname{div}\mathbf{V}_{ag}$$

as $\operatorname{div}\mathbf{V}_g$ vanishes apart from latitudinal changes.

Putting

$$\zeta = \operatorname{curl}_p \mathbf{V}_g = \operatorname{curl}_p \left(\frac{1}{l} \cdot l\mathbf{V}_g \right)$$

$$= \frac{1}{l} \operatorname{curl}_p (l\mathbf{V}_g) + l v_g \frac{\partial}{\partial x}\frac{1}{l} - l u_g \frac{\partial}{\partial y}\frac{1}{l}$$

and taking the y axis towards the north the second term on the right vanishes and the last becomes, where ϕ is the latitude and R the radius of the earth,

$$\frac{u_g}{l}\frac{\partial l}{\partial y} = \frac{u_g \cot \phi}{R}$$

This is the vorticity which would be due to the streamline curvature of a wind of velocity $u_g \cot \phi$ and radius of curvature equal to that of the earth and is small in the synoptic problem.

To this order of approximation therefore

$$\zeta = \frac{1}{l}\left(\frac{\partial}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial}{\partial y}\frac{\partial h}{\partial y} \right)_p = \frac{1}{l}\nabla_p{}^2 h$$

and equation (9) may be written

$$l \operatorname{div}_p \mathbf{V} = -(\mathbf{V} \cdot \nabla_p)(\zeta + l) - \frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p{}^2 h \qquad (10)$$

It will be noticed that the last term is virtually identical with the Brunt-Douglas isallobaric divergence term. The middle term is an advection term and the difference between them gives the actual divergence in quasi-geostrophic motion.

A condition for non-development

8. If a situation is non-developmental in the sense that $dp/dt = 0$ everywhere, $\operatorname{div}_p \mathbf{V}$ is also zero and

$$(\mathbf{V} \cdot \nabla_p)(\zeta + l) = -\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p{}^2 h \qquad (11)$$

It will also be observed that all the terms which were ignored in Section 6 as probably small involve dp/dt and therefore definitely vanish in the non-development case. Thus the above is an accurate condition for non-development subject only to the approximation due to the assumption of quasi-geostrophic motion.

It shows that in order to ensure zero development it is necessary that the local pressure field (or field of geopotential h) at every level shall so adapt itself as to keep in balance with the advection of a varying total vorticity—a conception which is altogether simple and reasonable. The local field of pressure is however the result not of motion at the level but of the whole three-dimensional motion and all levels are strictly connected by the hydrostatic relation.

We introduce this hydrostatic control by considering the difference in conditions between two chosen levels. Denoting the lower level by suffix o non-development requires that

$$(\mathbf{V} \cdot \nabla_p)(\zeta + l) - (\mathbf{V}_o \cdot \nabla_p)(\zeta_o + l) = -\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 (h - h_o)$$

$$= -\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h' \quad . \quad . \quad . \quad . \quad (12)$$

where h' is now the geopotential difference between the two pressure levels, the familiar "thickness" of upper-air analysis, and proportional to the mean temperature of the vertical air column.

In this form the criterion of non-development demands a balance between the changes in the temperature field and the vertical difference in vorticity as determined by the different rates of advection at different levels. It is apparent at once that in any natural situation there is no reason why the temperature field should be so modified as conveniently to fit in with the requirement of equation (12). There must therefore be development to some degree in all real situations and the difference in the divergence between any two levels is given approximately by

$$l\,(\mathrm{div}_p \mathbf{V} - \mathrm{div}_p \mathbf{V}_o) = -(\mathbf{V} \cdot \nabla_p)(\zeta + l) + (\mathbf{V}_o \cdot \nabla_p)(\zeta_o + l) - \frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h' \quad . \quad (13)$$

The developing situation

9. As argued in Section 4 the nature of the development can be inferred if the distribution with height of the relative divergence $\mathrm{div}_p \mathbf{V} - \mathrm{div}_p \mathbf{V}_o$ is known, but we see that this involves two distinct physical quantities. Firstly the change in the vertical distribution of vorticity as would be produced by shearing advection and secondly the variation in the temperature field

$$\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h' = \nabla_p^2 \left(\frac{\partial h'}{\partial t}\right)_p$$

—the "isallobaric" divergence obtained from the tendencies of the thickness topography.

$\left(\frac{\partial h'}{\partial t}\right)_p$ depends on the rate of change of mean temperature of the air column and may be produced in three ways—by isobaric advection, by adiabatic changes due to non-isobaric motion and by non-adiabatic heating or cooling processes.

Non-adiabatic effect

Relative upper divergence, from equation (13), is associated with a negative value of $\nabla_p^2 \left(\frac{\partial h'}{\partial t}\right)_p$ as with a dome in the isallobaric topography. Relative upper convergence is associated with a hollow in this topography. In so far as the effect is due to direct heating or cooling a region of local heating in the lower troposphere will tend to be cyclogenetic, local cooling anticyclogenetic. The result is of course well known and important.

Adiabatic effect

Subsidence (defined by the index dp/dt) always produces local isobaric temperature increase, ascent an increase or decrease according as the lapse rate is greater than or less than the adiabatic value (dry or saturated as appropriate). Observation shows that considered as contributions to $\partial h'/dt$ these effects are not generally negligible. They are however dependent on the development terms themselves, that is upon the terms in dp/dt which were ignored in Section 6. It is necessary therefore to consider them in connection with these neglected terms and it is thought to be unwise to base any inference upon them at this stage.

Advection effect

The advection changes in the field of temperature are regularly important and must be dealt with in some way before equation (13) can be interpreted, particularly since the other terms are also advectional.

Considering advection alone

$$\nabla_p^2\left(\frac{\partial h'}{\partial t}\right) = -\nabla_p^2\left(\bar{u}\frac{\partial h'}{\partial x} + \bar{v}\frac{\partial h'}{\partial y}\right)_p$$

where $\bar{u}, \bar{v}$ are the components of a mean velocity over the pressure interval.

With the quasi-geostrophic approximation

$$lv' = \frac{\partial h'}{\partial x}\,;\; lu' = -\frac{\partial h'}{\partial y}$$

and

$$-\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h = \nabla_p^2\,(\bar{u}v' - \bar{v}u')$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\bar{u}v' - \bar{v}u')$$

$$= \left(\bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}\right)\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right) + \left(\bar{u}\frac{\partial}{\partial y} - \bar{v}\frac{\partial}{\partial x}\right)\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)_p$$

$$-\left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y}\right)\left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right)_p - \left(u'\frac{\partial}{\partial y} - v'\frac{\partial}{\partial x}\right)\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)_p$$

Consistent with the quasi-geostrophic assumption and also with the order of approximation in equation (9) the terms in divergence must be ignored compared with those in vorticity and we may write, vectorially,*

$$-\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h' = (\bar{\mathbf{V}}\cdot\nabla_p)\,\zeta' - (\mathbf{V}'\cdot\nabla_p)\bar{\zeta}$$

A further approximation is obtained by assuming that over the pressure interval the direction of shear does not vary with height.

Then $\bar{\mathbf{V}} = \mathbf{V}_0 + a\mathbf{V}'$ where a is a scalar factor and

$$-\frac{1}{l}\left(\frac{\partial}{\partial t}\right)_p \nabla_p^2 h' = (\mathbf{V}_0\cdot\nabla_p)\zeta - (\mathbf{V}\cdot\nabla_p)\,\zeta_0$$

* Terms of the form $\frac{\partial \bar{u}}{\partial x}\cdot\frac{\partial v}{\partial x}$ etc. are omitted. They are connected with the processes of frontogensis and frontolysis by deformation and require separate attention.

Equation (13) then takes the simple form

$$
\begin{aligned}
l\,(\mathrm{div}_p\mathbf{V} - \mathrm{div}_p\mathbf{V}_o) &= -(\mathbf{V}\,.\,\nabla_p)\,(\zeta + l) + (\mathbf{V}_o\,.\,\nabla_p)\,(\zeta_o + l) + (\mathbf{V}_o\,.\,\nabla_p)\,\zeta \\
&\qquad - (\mathbf{V}.\nabla_p)\,\zeta_o \\
&= -[(\mathbf{V} - \mathbf{V}_o)\,.\,\nabla_p](l + \zeta + \zeta_o) \\
&= -(\mathbf{V}'\,.\,\nabla_p)\,(l + \zeta + \zeta_o) \\
&= -V'\,\frac{\partial}{\partial s}(l + \zeta + \zeta_o) \qquad \qquad (14)
\end{aligned}
$$

where $\frac{\partial}{\partial s}$ represents differentiation in the direction of the shear vector $\mathbf{V}'$.

A POSSIBLE APPLICATION TO FORECASTING TECHNIQUE

10. In the British Service at present the contours and "thickness" charts are regularly constructed for the levels 1,000 mb. (practically given by the m.s.l. isobars), 700 mb., 500 mb. and 300 mb. In so far as our expressions are valid it is therefore formally possible to determine $\mathrm{div}\mathbf{V} - \mathrm{div}\mathbf{V}_o$ for three layers and so to arrive at an indication of the relative divergence distribution.

With the quasi-geostrophic assumption the problem reduces to that of determining the quantity $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)h$ on the various charts, representing the results by isopleths and then estimating the values of the gradients along the shear vector.

The procedure implies some labour but it is formally a matter of routine computation from the geometry of the charts. Our theory may be over-simplified but the analysis seems to show that the quantities which we have derived are essential to any dynamical treatment. A different approach may of course lead to more easily computed functions but this hardly seems likely and it would seem justifiable to infer that, if the technique indicated above, based on approximations which must simplify the computations, is too laborious for routine work, then systematic application of dynamical methods is unattainable, a conclusion of despair.

The need for some dynamical or thermodynamical appreciation of the forecasting problem cannot be over-stressed. Present methods of general forecasting place great reliance on observing the developments indicated by successive charts; and extrapolating without much understanding of the physical processes. It is suggested therefore that it would be fully worth while to try out systematically the implications of the roughly quantitative formula derived above. The ideas have over a considerable period been used qualitatively, by general inspection of available charts, with some promise and it is hoped next to undertake a series of synoptic studies.

Working largely on extrapolation of observed tendencies supported by rather vaguely formulated rules of experience and some scientific generalisations the forecaster can at present produce with a good measure of success forecast charts for some 24 hours ahead showing the contours of the various surfaces. The forecast charts cannot however contain those elements, particularly the pressure tendency, upon which so much reliance is based in making the forecast and to that extent the forecast charts do not easily permit of further extrapolation. They do however provide precisely the material upon which estimates of development may be made by the method derived in this paper. To that extent therefore the method will afford at the same time a criterion of self-consistency between the predicted charts and an indication of the further developments not appearing on the current charts.

A FEW GENERALISATIONS

11. Pending the results of systematic investigation of actual situations a few general deductions from the theory are of interest.

The expression for relative divergence is

$$l\,(\mathrm{div}_p\mathbf{V} - \mathrm{div}_p\mathbf{V}_o) = -V' \frac{\partial}{\partial s}(l + \zeta + \zeta_o) \qquad (15)$$

$$= -V' \frac{\partial}{\partial s}\,(l + \zeta' + 2\zeta_o) \qquad (16)$$

As foreshadowed in Section 4 complex situations must be expected but in simple cases V', the shear, normally increases with height through a deep layer and then decreases in the stratosphere. Thus we may expect to obtain results of some general application by considering only two levels such as 1,000 mb. and 500 mb. Relative divergence indicates ascent and cyclogenesis, relative convergence the reverse. In this general discussion it is most convenient to discuss the expression in the form of equation (16) and in typical cases

$$-V' \frac{\partial}{\partial s}\,(l + \zeta' + 2\zeta_o) \gtrless 0$$

according as the development is of cyclonic or anticyclonic type (ascent or subsidence).

It is noted that generally development depends on V'—there is no development without shear—and development is likely to be most rapid in regions of strong shear or temperature gradient. In this respect the formula has fair promise.

The term $V' \dfrac{\partial l}{\partial s}$

This term would indicate that a shearing wind towards the poles should be associated with anticyclonic type of development and subsidence while shearing towards the equator would be associated with cyclonic type and ascent. The writer has not so far found the process particularly noticeable perhaps because the magnitude of the effect is not dominant in the more striking situations. We propose therefore to leave the question for the further evidence expected from systematic synoptic studies.

The term $2V' \dfrac{\partial \zeta_o}{\partial s}$ *—the thermal steering effect*

This term indicates simply that shearing over a surface maximum of vorticity implies cyclonic development ahead, anticyclonic to the rear, and the reverse distribution with a surface minimum of vorticity. Since broadly speaking a depression or trough is a region of maximum and an anticylone or ridge a region of minimum vorticity such features will tend to be displaced in the direction of shear by a process of development (as distinct from translation).

The effect is well known and provides working rules which are already part of the forecaster's equipment.

Considering this term alone

$$l\,(\mathrm{div}_p\mathbf{V} - \mathrm{div}_p\mathbf{V}_0) = -2V'\frac{\partial}{\partial s}\zeta_0$$

Integrating through the atmosphere and putting

$$\int_0^p \mathrm{div}_p\mathbf{V}\,.\,dp = 0$$

we get

$$l\,\mathrm{div}_p\mathbf{V}_0 = 2\overline{V}'\frac{\partial}{\partial s}\zeta_0$$

where $\overline{V}'$ is the mean shear wind velocity and $\frac{\partial}{\partial s}$ now means differentiating in this direction.

Referring now to equation (9) and ignoring the variation of l we get

$$\left[\frac{\partial}{\partial t} + (\mathbf{V}_0 + 2\overline{\mathbf{V}}')\,.\,\nabla\right]\zeta_0 = 0$$

showing that the development is equivalent to the translation of the vortex field by a velocity $2\overline{\mathbf{V}}'$ in addition to the natural speed V.

The term gives no development tendency at a region where $\frac{\partial\zeta_0}{\partial s}$ is zero—a maximum or minimum value of ζ_0 in the direction of shear. Since generally a centre of pressure is a maximum or minimum the shear development term does not tend to the local intensifying but only to the wave-like propagation of the feature.

The term $V'\frac{\partial}{\partial s}\zeta'$*—the thermal vorticity effect*

This quantity is determined entirely by the topography of the thickness chart and the distribution of thermal winds. It indicates broadly, cyclonic (ascent) type development where the thermal vorticity decreases in the direction of the shear, anticyclonic type (subsidence) where the thermal vorticity increases in the direction of shear (taking vorticity positive when cyclonic).

There are of course many possibilities but the most striking occur when the thermal field has marked (cold) troughs and (warm) ridges. Cyclonic development is a feature of the forward part of a cold trough or the rear of a warm ridge, anticyclonic development occurs on the other side in each case.

It is apparent at once that these implications fit in extremely well with known rules and principles. A depression in its early stages is often (by no means always) associated with a tolerably straight thermal pattern and moves by the direct steering effect—the term $V'\frac{\partial}{\partial s}\zeta_0$ is predominant. But at some stage the thermal pattern becomes distorted by the circulation, a cold trough develops in the rear and cyclonic type development is then over the surface centre—the term $V'\frac{\partial}{\partial s}\zeta'$ becomes important—and deepening not associated with movement is normal. The vigorous trough formations often occurring in the rear of depressions are regularly explainable on these grounds and in the author's experience are practically unknown without the co-existence of the cold upper trough.

The case of the normal occluding depression is however not the one which presents the greatest difficulty to the forecaster. Abnormal tracks, re-developments and secondary formations are far more troublesome and the consideration of the thermal vorticity field seems to give valuable insight. We may mention two common occurrences.

Firstly there is the case of the rapid retardation deepening and curving track of a depression which moves into a pre-existing thermal field forward of a cold tongue. This is a feature to be anticipated wherever, for any reason, a cold tongue appears on the charts.

Secondly there is the anticyclogenesis which appears when a warm tongue is produced by the circulation round a stagnating low. Anticyclones over Western Europe in winter may build up, when, owing to developments further west a depression is held up and advection allows a warm tongue to be formed over the eastern ocean. Then anticyclonic building is to be expected (or feared) and once started is self-generating for the circulation tends to maintain the warm tongue. The further processes of building followed by westward advance of cold continental air are all at least qualitatively explainable.

General comment on the results

12. Theoretical criticism of the results derived will rest primarily on the approximations used in the argument. The criticism may be serious but some defence is possible. The approximations are not fundamentally unreasonable. To use the geostrophic approximation for determining the magnitude of quantities like wind speed and vorticity while at the same time estimating therefrom quantities which vanish in strict geostrophic motion provides an obvious target for criticism but may still be entirely justified. The validity of an approximation depends on the magnitude of terms in the particular expression and divergence may be a negligible factor for some quantities and not for others.

Also it is claimed that the approximations are consistent and that we have not retained terms which have a tendency to be cancelled out by other terms ignored—a rather common failing in meteorological texts. It is fully realised that the terms ignored are at least in some situations significant. We know, for example, that the thermal pattern is much modified by adiabatic and direct heating and cooling processes and not merely by advection. We also know that the terms ignored in Section 6 involving dp/dt—in effect the vertical motion—are likely to be important in vigorous development and indeed the writer, in association with Durst (1938), has stressed the significance of the vertical motion. But in spite of this it is felt that we have extracted from a complex problem a tractable expression which is likely to represent an important contribution to the development process. It is the hope that experience will allow the working forecaster to develop empirical adjustments to the inferences which this partial theory provides and so arrive at something better than almost pure empiricism.

It will be observed that the general deductions from the term $V' \frac{\partial}{\partial s} \zeta'$ come to much the same thing as the arguments used by Bjerknes (1937, 1940) in relating the deepening of extra tropical cyclones with the upper cyclonic and anticyclonic flow patterns due to the movement of the upper air through the patterns. The term $V' \frac{\partial}{\partial s} \zeta_0$ suggests an explanation of the common distribution of ascent and subsidence in the neighbourhood of depressions and troughs, anti-cyclones and ridges as well known to synoptic meteorologists.

Thus it may be fair to assume that we have the basis for a method of dealing on a routine basis with processes which do occur although they are not the complete story. Since the arguments and deductions are susceptible both to physical interpretation and to practical test they may have some acceptable virtue.

It is however necessary not to read too much into the argument. The treatment attempts to diagnose where development (subsidence or ascent) is to be expected. Although for convenience subsidence in the lower atmosphere may be called anticyclonic type of development and ascent cyclonic as being associated with the production of anticyclonic or cyclonic vorticity at low levels it must not be inferred that this necessarily means the formation of an anticyclone or depression or even a region of positive or negative isallobars although this may be usual.

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