EAS 471 Assignment 3 2007

Eulerian Simulation of Dispersion in the stably-stratified hhASL

Write a program to calculate the mean concentration field $C = C(x, z)$ downwind from a continuous line source[1] at $x = 0, z = h_s$ in the horizontally-uniform, stably-stratified atmospheric surface layer. Assume C is the solution of

$$U(z)\,\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left[K(z)\,\frac{\partial C}{\partial z}\right] \tag{1}$$

and that the surface layer mean windspeed and eddy diffusivity profiles are

$$U(z) = \frac{u_*}{k_v}\left[\ln\frac{z}{z_0} + \beta\,\frac{z - z_0}{L}\right] \tag{2}$$

$$K(z) = \frac{(k_v/S_c)\,u_*\,z}{1 + \beta z/L} \tag{3}$$

where the Obukhov length $L > 0$, $k_v = 0.4$ is the von Karman constant, the empirical factor $\beta = 5$, and S_c is the turbulent Schmidt number.

Discretize using grid-lengths $\Delta x \sim \Delta z \sim 0.1$ m. For two choices $S_c = (0.63,\ 1.0)$ of the Schmidt number, compare your calculated solution $C(100, z)$ at $x = 100$ m with the observations (Table 1) of Project Prairie Grass run 59, for which source height $h_s = 0.46$ m, and the meteorological situation was: $u_* = 0.14$ m s^{-1}, $L = 7$ m, $z_0 = 0.005$ m. Alongside your solution, plot also the solution for $|L| = \infty$ (neutral stratification).

[1] By symmetry, this is identical to the crosswind-integrated concentration at the same distance downwind from a continuous point source. This is the situation for which we have experimental data, viz., Project Prairie Grass.

Method

The algorithm suggested in class is of the form:

$$A^C_{I,J}\, C_{I,J} = A^N_{I,J}\, C_{I,J+1} + A^S_{I,J}\, C_{I,J-1} \; + B_{I,J} \tag{4}$$

where the $A_{I,J}$ are the centre (C), north (N) and south (S) "neighbour coefficients", and $C_{I,J}$ is the concentration matrix. But in fact since our neighbour coefficient vectors $A^C_{I,J}, A^N_{I,J}, A^S_{I,J}$ do not change with I we can simplify the notation to

$$A^C_J\, C(I,J) = \; A^N_J\, C(I,J+1) + \; A^S_J\, C(I,J-1) + \; B_{I,J}, \quad J = 1..J_{mx} \tag{5}$$

This is a marching problem, implicit along the J (vertical)-axis. You will need to use a Tridiagonal Matrix Inversion Algorithm, eg. that given in *Numerical Recipes* or the one in the Appendix at the end of this document.

Upper and lower boundary conditions

Set the top of your domain $z(J_{max})$ sufficiently high (say, at least 15 m) that $C(J_{max}) = 0$. Then (using the terminology of the TDMA subroutine given in the Appendix), your coefficients at $J = J_{max}$ are

$$\begin{aligned} a(J_{max}) &= A^C_{J_{max}} = 1 \\ b(J_{max}) &= A^N_{J_{max}} = 0 \\ c(J_{max}) &= A^S_{J_{max}} = 0 \\ d(J_{max}) &= 0 \end{aligned} \tag{6}$$

If we presume our gas does not react with the ground, we want zero flux

to ground, which is assured by requiring $C(I,1) \equiv C(I,2)$. Thus at $J = 1$ your coefficients are

$$\begin{aligned} a(1) &= 1 \\ b(1) &= 1 \\ c(1) &= 0 \\ d(1) &= 0 \end{aligned} \tag{7}$$

Inlet boundary condition

Now note that the governing equation is homogeneous... how is your solution going to "know" there is a source? Let J_h be the height index of the cell the physical source will lie within, and let the streamwise index value $I = 0$ (or 1 if you prefer) correspond to gridpoints aligned at the source location. The easiest approach is to set the inlet or inflow concentration profile as

$$C(0,J) = \begin{cases} 0 & \text{if } J \neq J_h, \\ 1/\left(U(J_h)\,\Delta z\right) & \text{if } J = J_h \end{cases} \tag{8}$$

which guarantees that the total mass flux across the first interior plane $I = 1$ will be

$$Q = \sum_J C(0,J)\,U(J)\,\Delta z = 1 \tag{9}$$

Last words

Provide a flowchart that, in conjunction with your table of symbols, unambiguously defines your algorithm. Experiment with the sensitivity of the computed concentration profile to gridlengths Δx, Δz.

Before solving the diffusion problem, make sure your implementation of the matrix inversion program correctly solves a simple test problem; eg. suppose the problem is of form $\mathbf{M}\ \mathbf{S} = \mathbf{R}$ where the coefficient matrix is $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ ... you should be able to apply TDMA to determine that $\mathbf{S} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Table 1: Normalized, crosswind-integrated concentration $z_0 u_* C/(k_v Q)$ observed at distance $x = 100$ m from the source (height $h_s = 0.46$ m) in Project Prairie Grass run 59.

z[m]	$z_0 u_* C/(k_v Q)$
7.5	7 E-7
4.5	2.1E-5
2.5	1.05E-4
1.5	1.75E-4
1.0	2.14E-4
0.5	2.40E-4

Appendix: TDMA algorithm

```
c
      subroutine tdma(m,n,a,b,c,t,d)
c
c     structure is:  a(j) t(j) = b(j) t(j+1) + c(j) t(j-1) + d(j)
c
c     where vector t(j), j=m...n is the solution we compute from
c     our neighbour coefficients (a=centre, b=north or  upper
c                                 c=south or lower) and from our
c     and the coefficient vector that depends on the solution
c     "behind our back", d(j)
c
c     don't confuse the local coefficient c(j) with the
c     concentration field C(I,J) that we are seeking !
c
      implicit none
      integer nx,ny,nz,span
      parameter (span=125)        ! adjust as appropriate
      integer j,k,m,n
      real denom
      real a(span),b(span),c(span),d(span),t(span)
      double precision p(span),q(span)
      p(m)=b(m)/a(m)
      q(m)=d(m)/a(m)
      do j=m+1,n
        denom=a(j)-c(j)*p(j-1)
        p(j)=b(j)/denom
        q(j)=(d(j)+c(j)*q(j-1))/denom
      end do
      t(n)=q(n)
      do k=1,n-m
        j=n-k
        t(j)=p(j)*t(j+1)+q(j)
      end do
      return
      end
c
```