EAS471, Lab 2 "The Heat Equation" Feb, 2007

Task

Solve numerically the heat equation

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \tag{1}$$

on the domain $0 \leq x \leq L$, with initial and boundary conditions

$$\begin{aligned} T(x,0) &= 0 \\ \left(-K \frac{\partial T}{\partial x}\right)_{x=0} &= Q \\ \left(-K \frac{\partial T}{\partial x}\right)_{x=L} &= 0 \end{aligned} \tag{2}$$

where K [$\mathrm{m^2\,s^{-1}}$] is the thermal diffusivity (note: italicized, to avoid confusion with the Kelvin unit K) and Q [$\mathrm{K\,m\,s^{-1}}$] is a kinematic heat flux forcing the solution.

Use the explicit, conditionally stable discretization

$$T_I^{n+1} = \gamma\, T_{I-1}^n + (1-2\gamma)\, T_I^n + \gamma\, T_{I+1}^n \tag{3}$$

that results from choosing a forward time-difference for $\partial T/\partial t$ and the usual molecule for the diffusion term (γ is the diffusion number, given below).

Express your solutions in terms of dimensionless variables using appropriate scales for length, temperature, time, etc., eg. the temperature scale is

QL/K and

$$\begin{aligned}
\phi(\xi, t^*) &= T \, \frac{K}{QL} \\
t^* &= \frac{t\,K}{L^2} \\
\xi &= \frac{x}{L}
\end{aligned} \tag{4}$$

and bear in mind that it might make sense to reframe the problem in terms of the dimensionless variables and solve *directly*, ie. formulate your algorithm directly in terms of $\phi(\xi, t^*)$ as function of ξ, t^* where $0 \leq \xi \leq 1$ (note: this amounts to setting $Q = K = L = 1$ in the above formulation, with the result that the natural timescale for the problem becomes unity).

Experiment with a range of values for the timestep Δt and gridlength[1] Δx, so as to test the relevance of the theoretical stability criterion

$$\gamma = \frac{K\,\Delta t}{\Delta x^2} \equiv \frac{\Delta t^*}{\Delta \xi^2} \leq \frac{1}{2} \tag{5}$$

on the diffusion number γ.

In your write-up please show graphs of your solution $\phi^{num}(I, n)$ versus $\xi(I)$ out to time $t^* = n\Delta t^* \leq 5$ (ie. several times the natural timescale for equilibration of the solution) to show the time evolution, and compare with the analytical solution[2]

$$\phi^{aly}(\xi, t^*) = t^* + \frac{1}{2}\,(\xi - 1)^2 - \frac{1}{6} - \frac{2}{\pi^2} \sum_1^{\infty} \frac{1}{m^2} \exp\left(-m^2 \pi^2 t^*\right) \cos\left(m\pi\xi\right) \tag{6}$$

[1] You want a gridpoint at each end of the interval (0,L), and equal intervals between gridpoints; so if the gridpoints are labelled $i = 1...N$ then $\Delta x = L/\mathrm{real}(N-1)$. Now having established your Δx you compute your timestep as $\Delta t = \gamma\,\Delta x^2/K$, or (in the dimensionless formulation) $\Delta t^* = \gamma\,\Delta \xi^2$.

[2] Thanks to Dr. Gordon Swaters, Mathematics, UA, for assistance in obtaining this solution.

(you can truncate the sum at $m = M$; play with M, eg. $M = 10$ or $M = 50$ or $M = 100$). Knowing the latter permits to determine the discretization error (ie. difference between numerical solution and analytical solution), thus, plot $|\,\phi^{aly} - \phi^{num}\,|$ at one or more gridpoints for fixed t^*, as function of $\Delta\xi$, and comment in relation to the known truncation error of our numerical procedure[3].

[3]Note added after marking the assignments: it would have been useful to suggest that you plot $|\,\phi^{aly} - \phi^{num}\,|$ at fixed ξ, t^* in two ways: (a) as function of $\Delta\xi$ for fixed Δt_*; and (b) as function of Δt_* for fixed $\Delta\xi$. In each case, γ would vary. However this would have enabled to determine whether the discretization error had the expected functional dependence on $\Delta t^*, \Delta\xi$