The “relaxation” method for solving elliptic differential equations

(EAS 471. Updated March 28, 2007)

Consider the steady-state diffusion equation

$$\frac{\partial T}{\partial t} = 0 = \kappa \nabla^2 T + Q \tag{1}$$

where (for example) κ is thermal diffusivity and $Q = Q(x, y, z)$ is an arbitrary but steady source. Knowing the action of the Laplacian operator is to smooth out sharp spatial curvature, our intuition suggests the solution should be a field $T(x, y, z)$ that is as smooth as can be consistent with the source distribution Q and the boundary conditions, both here unspecified.

It is also a reasonable guess that if we started out with the *wrong* solution and plugged it into the time-dependent equation

$$\frac{\partial T}{\partial t} = R = \kappa \nabla^2 T + Q \tag{2}$$

then any stable numerical solution technique that is consistent with the differential equation should nudge our solution towards the correct steady state solution... and when we reach that solution the “residual” $R(x, y, z)$ will vanish.

Let us specialize to 2 space dimensions, and discretize as

$$\kappa \, \frac{T^n_{i+1,j} + T^n_{i-1,j} - 2T^n_{i,j}}{\Delta x^2} + \kappa \, \frac{T^n_{i,j+1} + T^n_{i,j-1} - 2T^n_{i,j}}{\Delta y^2} + Q_{i,j} = R^n_{i,j} \tag{3}$$

where $T^n_{i,j}$ is the n$^{\text{th}}$ trial field (ie. not the correct solution; if $n = 1$ it is our first guess field). Now, any non-zero value of the residual R^n_{ij} indicates that we don’t yet have a valid solution, at least at this grid point. However we can adjust the value of T at the local gridpoint to make the (new) residual exactly zero by making an adjustment

$$T^{n+1}_{i,j} \leftarrow T^n_{i,j} + \alpha \, R^n_{i,j} \tag{4}$$

Substituting this value in eqn. (3):

$$\begin{aligned} \kappa \, \frac{T^n_{i+1,j} + T^n_{i-1,j} - 2\left(T^n_{i,j} + \alpha R^n_{i,j}\right)}{\Delta x^2} &+ \kappa \, \frac{T^n_{i,j+1} + T^n_{i,j-1} - 2\left(T^n_{i,j} + \alpha R^n_{i,j}\right)}{\Delta y^2} + Q_{i,j} \\ &= R^n_{i,j} - 2 \, \kappa \, \alpha \, R^n_{i,j} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \\ &= R^{n+1}_{i,j} \end{aligned} \tag{5}$$

But we may choose α to make the $R_{i,j}^{n+1}$, viz:

$$\alpha = \frac{1}{2\,\kappa}\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)^{-1} \equiv \frac{1}{2\,\kappa}\,\frac{\Delta x^2\,\Delta y^2}{\Delta x^2+\Delta y^2} \tag{6}$$

Of course, the moment we change $T_{i,j}$ we affect the residual not only locally but at neighbouring points. However it turns out that if one simply iterates across the whole grid, repeatedly, in any order, the residuals progressively get smaller as the iteration count n increases.

One needs a criterion for cessation of iterations. How small must the residual $R_{i,j}$ be driven? A typical approach is to demand that the root mean square residual

$$\sigma_R = \left((I_{mx}\,J_{mx})^{-1} \sum_{i=1}^{Imx} \sum_{j=1}^{Jmx} R_{i,j}^2 \right)^{1/2} \tag{7}$$

be smaller than some criterion ϵ, where ϵ is chosen on physical grounds.

In the context of our baroclinic prog, the source term $Q_{i,j}$ is the vorticity $\zeta_{i,j}$ and the variable we seek is the streamfunction ψ, viz.

$$\nabla^2\,\psi = \zeta \tag{8}$$

We will require a subroutine that implements the relation method to solve for ψ, given its value on the boundary, the forcing ζ, and the limit ϵ. The call will look something like:

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call relax(imin,jmin,imax,jmax,psi,zeta,epsilon)
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where on entry ψ is the first guess field and must have the correct values on the boundaries ($i = imin$ or $i = imax$ or $j = jmin$ or $j = jmax$), values which will not be changed by the subroutine. How small should epsilon be? It has units of velocity shear, so a millimetre per second over 1000 m is a small value, say, $\epsilon = 10^{-6}\,\mathrm{s}^{-1}$