

Eulerian Derivation of a Generalised Conservation Equation

We'll derive a generalized conservation equation for two-dimensional flow - the generalization to three dimensions is mechanical.

Let c denote the amount of a certain property per unit volume of the fluid, eg. if we have a 'mass' property c has the units $[\text{kg m}^{-3}]$. Consider a "control volume" (cv), an imaginary surface (see Fig. 1) through which the fluid freely moves, having dimensions $\Delta x, \Delta z$. Both diffusion and convection can transfer " c " across the control volume (cv) walls. Through all space, the property is created at a rate Q $[\text{kg m}^{-3} \text{ s}^{-1}]$.

At any time the amount in the cv is $c(t)\Delta x\Delta z$ (where c and Q are for now considered to be average values throughout the small control volume). This changes over time Δt by an amount $\Delta c \Delta x \Delta z$ due to internal production and net losses across the walls.

Let F_x, F_z be respectively the total (convective + diffusive) flux components of c along x and along z , $[\text{kg m}^{-2} \text{ s}^{-1}]$. Then:

$$\begin{aligned}\Delta c\Delta x\Delta z &= \Delta z\Delta t [F_x(x) - F_x(x + \Delta x)] \\ &+ \Delta x\Delta t [F_z(z) - F_z(z + \Delta z)] + Q\Delta x\Delta z\Delta t\end{aligned}\quad (1)$$

where the fluxes are considered average values along the (small) faces. Now divide by $\Delta x\Delta z\Delta t$ and let $(\Delta x, \Delta z, \Delta t)$ all become infinitesimal. We obtain the differential equation:

$$\frac{\partial c}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_z}{\partial z} + Q\quad (2)$$

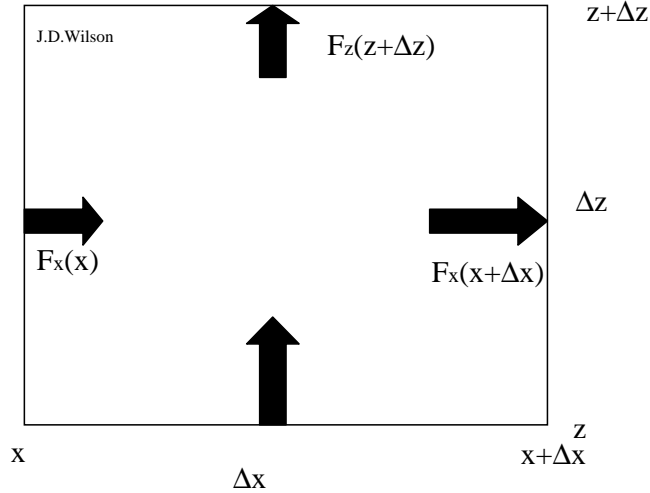


Figure 1: *An imaginary control volume through which the fluid flows freely, and the interfacial fluxes of “c” carried by the flow. Arrows indicate the direction of positive flux.*

Generalising to 3d, we have:

$$\frac{\partial c}{\partial t} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} + Q \quad (3)$$

which may also be written in these two following elegant and coordinate-independent forms:

$$\frac{\partial c}{\partial t} = -\frac{\partial F_i}{\partial x_i} + Q \quad (4)$$

and

$$\frac{\partial c}{\partial t} = -\nabla \cdot \vec{F} + Q \quad (5)$$

In words, the local rate of change in time (local tendency) of the concentration of a fluid attribute is driven by the divergence of the flux of that attribute, and/or internal production.

The LHS is a “storage term.” The first term on the right is a “transport term” since it is of the form of a spatial derivative. And the second term on the right is (obviously) a “source term.” The transport equations of fluid mechanics contain only these three types of terms.

Holton (2004) derives specifically the conservation equation for air mass (the “continuity equation”), ie. $c \rightarrow \rho$. “Air” is neither created nor destroyed in situ, so there is no Q term; and because air does not diffuse in air, there is no diffusive flux and so the flux of air is purely convective: $\vec{F} = \vec{u} \rho$.

References

Holton, J.R. 2004. *An Introduction to Dynamic Meteorology (Fourth Edition)*. Elsevier.