Atmospheric radiation: supplement replaces Ch.12 of EAS 471 Notes (April, 2007)

Update: April 14, 2007

Note: Here I have exchanged the roles of θ, ϕ relative to notation used in the classroom, so as to be consistent with the textbooks referred to.

Preamble: In the mid-latitude cloudless summer atmosphere, direct atmospheric heating due to solar absorption is of order $+2\ K$ day^{-1}, while the long wave cooling rate is of order $-2\ K$ day^{-1}. This would hardly recommend parameterizing direct radiative heating in the context of short range weather forecasting, however, much stronger local heating/cooling rates may occur in cloudy layers. And in the context of climate models, it is essential to develop confidence that "cloud feedbacks" have been included with fidelity: according to Mitchell (2004) "Most of the range in (modelled) climate sensitivity is associated with differences in cloud feedback", and the overall cloud feedback ranges from being moderately negative to being strongly positive, among the various climate models.

Difficulties with radiative transfer are principally computational. There exist several very different computational techniques proven to give a good description of radiative scattering and absorption. However since in principle for greatest accuracy a spectral description is needed (ie. one calculates the radiation-matter interactions in detail for each of a larger number of very narrow wavebands— so called "line by line" methods), the most rigorous approaches are too cumbersome to be employed in weather/climate models. Widely differing approaches to computing radiation energy transfer stem from differences in the system under study, in objectives, and in computational resources.

Present parametrizations typically employ "band" models, ie. it is assumed that one may with suitable degree of accuracy define mean optical coefficients for scattering and absorption over a selected set of spectral intervals. Due to the manifold possible interactions of photons of various frequencies with molecules or particles/droplets in the atmosphere, it is necessary to employ finer subdivisions of the spectrum than simply the familiar dichotomy: shortwave (solar, about $0.1 - 4$ μm) versus longwave (terrestrial, about $4 - 100$ μm).

Definition of "Intensity" of radiative transfer in a given waveband

Radiative transfer is omnidirectional - photons are flying about in any and every direction. The fundamental descriptor of the radiation field (see Fig. 1) is the intensity $I(\mathbf{x}, \mathbf{s})$, with units [J s^{-1} m^{-2} steradian^{-1}], defined more specifically to be the intensity of energy transfer at position $\mathbf{x}$ into a cone of unit solid angle about the direction $\mathbf{s}$ (in a spectral treatment, the "spectral" or "monochromatic" intensity I_λ would have units [J s^{-1} m^{-2} μm steradian^{-1}]). For example, I might be the diffuse solar intensity; or the longwave intensity. (See also Liou, 2002; p4, Fig. 1.3)

The direction $\mathbf{s}$ is characterised by a unit vector ($\mathbf{s}$), and relative to Cartesian axes (see Fig. 1) aligned with the local zenith (unit vector $\hat{k}$), $\mathbf{s}$ has components

$$\mathbf{s} = \hat{\imath}\cos\phi\sin\theta + \hat{\jmath}\sin\phi\sin\theta + \hat{k}\cos\theta \tag{1}$$

The *net* energy flux density[1] across a unit of area whose normal is oriented along the arbitrary unit vector $\mathbf{d}$ is given by[2]

$$F_{\mathbf{d}}\,(\mathbf{x}) = \int_{4\pi} I(\mathbf{x},\mathbf{s})\;(\mathbf{d}\bullet\mathbf{s})\;d\omega \tag{2}$$

where $d\omega = \sin\theta\; d\phi\; d\theta$. Thus for example the *net* vertical radiant energy flux density (due to all photons within the waveband in question, ie. which contribute to the intensity I) is:

$$F_z(\mathbf{x}) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I(\mathbf{x},\phi,\theta)\;\cos\theta\;\sin\theta\; d\phi\; d\theta \tag{3}$$

while

$$F_z\uparrow(\mathbf{x}) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I(\mathbf{x},\phi,\theta)\;\cos\theta\;\sin\theta\; d\phi\; d\theta \tag{4}$$

is the hemispheric (1-way, upward) flux, in Liou's terminology, the "irradiance" from the lower into the upper hemisphere. Now, clearly if we define

$$F_z\downarrow(\mathbf{x}) = \int_{\phi=0}^{2\pi}\int_{\theta=\pi/2}^{\pi} I(\mathbf{x},\phi,\theta)\;\cos\theta\;\sin\theta\; d\phi\; d\theta \tag{5}$$

then

$$F_z \equiv F_z\uparrow + F_z\downarrow \tag{6}$$

[1]By convention (eg. see Liou, 2002, p6) a radiative flux has units [W] while more commonly we encounter the "radiant flux density" [W m^{-2}], an example of which is the solar constant.

[2]Note that unit vectors don't carry units: thus $\mathbf{d}\bullet\mathbf{s}$ does not, as would appear on first sight, contribute an unwanted [m^2].

However it is more common to define the net flux as one or other of

$$F_z = F_z \uparrow - F_z \downarrow \tag{7}$$

$$F_z = F_z \downarrow - F_z \uparrow \tag{8}$$

which (if we take the first choice) implies that we define

$$\begin{aligned} F_z \downarrow (\mathbf{x}) &= \int_{2\pi} I(\mathbf{x}, \mathbf{s}) \; (-\mathbf{k} \bullet \mathbf{s}) \; d\omega \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=\pi/2}^{\pi} I(\mathbf{x}, \phi, \theta) \; \cos\theta \; \sin\theta \; d\phi \; d\theta \end{aligned} \tag{9}$$

where $\mathbf{d} = -\mathbf{k}$ is a *downward* pointing unit vector.

Similar expressions obtain for the net fluxes in the orthogonal directions F_x, F_y, and the heating rate (again, due to photons of this waveband) is given by

$$h \left[\mathrm{W\ m^{-3}}\right] = -\nabla \bullet \mathbf{F} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} - \frac{\partial F_z}{\partial z} \tag{10}$$

(an identical expression holds for the monochromatic heating rate h_λ, which would carry units $\mathrm{W\,m^{-3}\,\mu m^{-1}}$). Thus if we can compute the intensity $I(\mathbf{x}, \mathbf{s})$, we can derive all the quantities of interest.

Note that eqn. (10) is valid provided on each coordinate axis positive net flux is defined as transferring energy in the positive direction.

Emission

Preamble: The "black body spectral (or 'monochromatic') emissive power" (Sparrow & Cess, 1978) is defined as the energy emitted (by a black body)

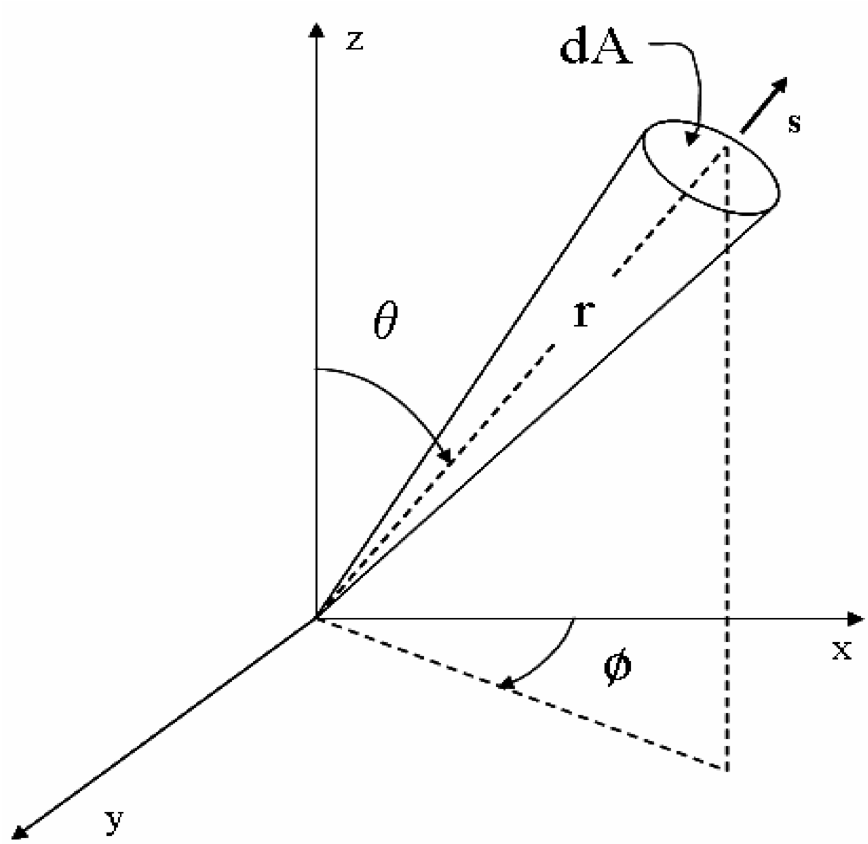

Figure 1: Definitional diagram for the radiative intensity I in direction $\mathbf{s}$. As defined here, the intensity I is a broadband property (though a spectral intensity I_λ can be defined similarly), and has units [J s^{-1} m^{-2} steradian^{-1}]. The solid angle defined by the cone is $d\omega = dA/r^2$, while the element of area is $dA = (r \sin\theta \; d\phi) \, r \; d\theta$. (Note that Liou, 2002 gives a slightly more complex definition, in which the element of area at the entry point to the cone has its normal parallel to the z-axis rather than parallel to $\mathbf{s}$.)

per unit of time and per unit of body surface area into the facing hemisphere (solid angle 2π) per unit of wavelength[3], and is given by Planck's law as

$$e_\lambda(T) = \frac{2\pi h c^2}{\lambda^5 \ (e^{hc/(k_B\,\lambda\,T)} - \ 1)} \tag{11}$$

(h = 6.626 x 10^{-34} J s^{-1} Planck's constant; k_B = 1.3806 x 10^{-23} J K^{-1} Boltzmann's constant; $c = n\,c_0$ the speed of light in the medium, where n is the index of refraction and c_0 is the speed of light in vacuum). A black body is an isotropic emitter (spectral intensity independent of direction $\mathbf{s}$ relative to the normal to the surface), but to obtain the spectral intensity I_λ [J s^{-1} m^{-2} μm steradian^{-1}]) implied by the emissive power (eqn. 13) does not (as one might on first sight expect) entail a division by 2π, but by π (this is explained below). Thus, the 'Planck function' is

$$B_\lambda(T) = \frac{e_\lambda(T)}{\pi} = \frac{2 h c^2}{\lambda^5 \ (e^{hc/(k_B\,\lambda\,T)} - \ 1)} \tag{12}$$

Now for an absorbing, emitting and scattering medium in thermal equilibrium, Kirchoff's law relates the rates of absorption and emission of photons by the medium, and gives that the volumetric energy emission rate J_λ is

$$J_\lambda = \frac{\rho\,\kappa_\lambda}{\pi}\ e_\lambda(T) = \rho\,\kappa_\lambda\,B_\lambda(T)\ [\mathrm{J\,s^{-1}\,\mu m^{-1}\,m^{-3}\,steradian^{-1}}] \tag{13}$$

where κ_λ is the spectral absorption coefficient (defined below), such that $[\rho\,\kappa_\lambda]$=[m^{-1}].

[3] Sparrow and Cess formulate per unit of frequency; the conversion is straightforward, since $e_{b\nu}\,d\nu = -\,e_{b\lambda\,d\lambda}$.

Note that below we shall encounter spatial gradients in e_λ or the related functions. These link to temperature gradients, for

$$\nabla e_\lambda = \frac{\partial e_\lambda}{\partial T}\, \nabla T \tag{14}$$

where

$$\frac{\partial e_\lambda}{\partial T} = \frac{2\pi h^2 c^3}{k_B\, \lambda^6\, T^2}\, e^{hc/(k_B\, \lambda T)}\, \left(e^{hc/(k_B\, \lambda T)} - 1\right)^{-2} \geq 0 \tag{15}$$

Spectral Radiative Transfer Equation

The radiative transfer equation to be given here will put a particular emphasis on the variation of intensity along a particular coordinate, z. It is therefore appropriate for the situation of a horizontally-uniform (ie. 'plane parallel') atmosphere, and not the most general possible statement one might derive.

When radiation of incident intensity $I_\lambda(\mathbf{s})$ traverses a finite "generalized distance" da of medium, we may distinguish these modes of change:

- extinction due to absorption, *modeled* as: $dI_\lambda = -\kappa_\lambda\, I_\lambda\, da$
- extinction due to scattering (away from direction $\mathbf{s}$), which decreases intensity, *modeled* as: $dI_\lambda = -\gamma_\lambda\, I_\lambda\, da$
- scattering *into* the direction $\mathbf{s}$, which increases intensity
- emission into direction $\mathbf{s}$, which increases intensity

(the two mechanisms for gain in intensity are sometimes combined into an abstract 'source function'). Here $\kappa_\lambda, \gamma_\lambda$ are the monochromatic absorption and scattering coefficients, and their sum

$$\beta_\lambda = \kappa_\lambda + \gamma_\lambda \tag{16}$$

is the monochromatic extinction coefficient. The increment da measures the "amount of matter" per unit area encountered over the path across which dI occurs. If $d\ell$ is an increment in distance[4] and ρ is the partial density (of the gas of interest), then if we define $da = \rho d\ell$ [kg m^{-2}], then β_λ [m^2 kg^{-1}] is called the "mass" extinction coefficient (or mass extinction cross section; eg. Liou, p27, whose k_λ is a monochromatic mass extinction coefficient); while if $da = d\ell$, β_λ is called the "volume" extinction coefficient. The product $\beta_\lambda\, da$ is dimensionless, and it can be defined to be equal to the increment in optical path[5]

The RTE we are about to derive is a differential equation of first order. It governs the intensity $I_\lambda(\mathbf{x}, \mathbf{s})$. The axis defined by the direction $\mathbf{s}$ is 'one-way' in the sense that the flux of photons flying in the direction of positive $\mathbf{s}$ originates on one side. Thus in our derivation of the RTE we depart from some authors, and decompose the intensity $I_\lambda(\mathbf{x}, \mathbf{s})$ into two sub-components such that

$$I_\lambda(\mathbf{x}, \mathbf{s}) \equiv I_\lambda^+(\mathbf{x}, \mathbf{s}) - \; I_\lambda^-(\mathbf{x}, \mathbf{s}) \tag{17}$$

[4]There is a subtlety here: as observers we will focus on a certain volume of the atmosphere, perhaps a slab, and assign a depth dz or a slant distance (eg. $dz/\cos\theta_0$, where θ_0 is solar zenith angle, zero if the sun is at the zenith) across the slab; but an electromagnetic wave (or photon) traversing our slab may experience multiple reflection and so, in a sense, take a much longer path. Thus it seems to me we have to think of our $d\ell$ as having to do with our mapping, not to do with distances travelled by photons.

[5]It is important to bear in mind that, in general, a distinction is made between an optical "path" and an optical "depth".

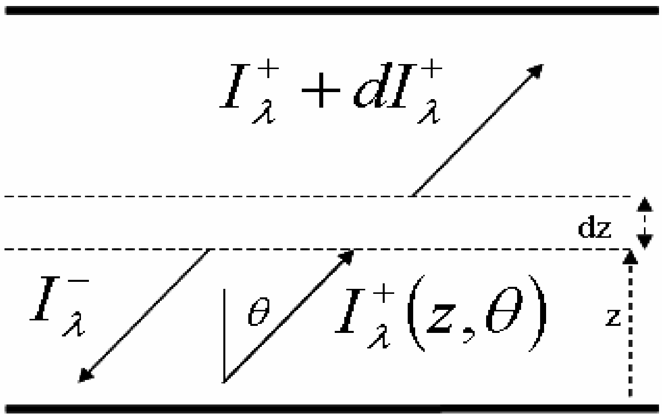

Figure 2: Definitions in regard to the upward component (+) of the radiative spectral intensity I_λ^+, and its variation with position along direction θ across an increment of the z axis.

(see Fig. 2). Note that our approach here is auspicious for application in the atmosphere, where in the simplest problems our focus is on changes on the height axis, with symmetry assumed on horizontal planes ('plane parallel atmosphere'). More generally, there is no reason for the z axis to be given the special status awarded by our derivation.

The change dI_λ^+ is (most generally) composed of these parts

- loss (extinction) due to (the sum of) absorption and scattering

$$dI_\lambda^{+,\,\text{extinction}} = -\beta_\lambda\, \rho\, I_\lambda^+\, dz/\cos\theta \tag{18}$$

- increase $J_\lambda = \kappa_\lambda\, B_\lambda(T)$ per unit volume due to monochromatic emission. Let A be a unit of area normal to z, $A\,dz$ an increment of volume. We can compute the increase in intensity per unit of area normal to I_λ^+ as $\kappa_\lambda\, \rho\, B_\lambda(T)\; A\,dz(A\,\cos\theta)^{-1}$,

$$dI_\lambda^{+,\,\text{emission}} = \kappa_\lambda\, \rho\, B_\lambda(T(z))\, dz/\cos\theta \tag{19}$$

- gain due to scattering *into* the cone of solid angle. This term entails an integration over all entry angles ϕ', θ' and involves the scattering coefficient and the scattering cross-section (defined below). Sparrow and Cess (1978, p209) give an expression valid for the special case of coherent, isotropic scattering. However we shall delay introducing any more specific formulation and simply represent this term symbolically by $G_{\lambda s}^+$ (gain by scattering)

Note that it is implicit that this is a continuum description, applicable if the increment in path is large relative to the sizes of molecules, aerosols, cloud droplets, and anything else interacting with radiation.

Combining the above definitions we obtain the radiative transfer equation(s), in the spectral form given here known also as the Schwarzschild equation (c.f. Liou, 2002, eqn1.4.5; Sparrow and Cess, 1978, eqns. 7-11,7-

12)

$$\mu \frac{dI_\lambda^+(z,\theta)}{dz} = -\beta_\lambda \rho I_\lambda^+(z,\theta) + \kappa_\lambda \rho B_\lambda(z) + G_{\lambda s}^+$$
$$\mu \frac{dI_\lambda^-(z,\theta)}{dz} = -\beta_\lambda \rho I_\lambda^-(z,\theta) + \kappa_\lambda \rho B_\lambda(z) + G_{\lambda s}^- \qquad (20)$$

where $\mu = \cos\theta$. Note that due to the use of 'mass' coefficients (to which extent I differ from Sparrow and Cess), $(\beta_\lambda \rho)$ and $(\kappa_\lambda \rho)$ must have units of the reciprocal of length.

Bulk RTE

The RTE is commonly given for a spectral band as

$$\mu \frac{dI}{d\tau} (= \frac{dI}{d\tau_0}) = -I + J_{em,sc} \qquad (21)$$

with appropriate interpretation of units. Here the vertical coordinate τ is the 'optical path' (see Fig. 3), defined such that its increment $d\tau \equiv \beta\rho dz$ (β is the bulk extinction coefficient, and $d\tau/\mu = d\tau_0$, the slant optical path). The first term on the right is the extinction term, and the second the source term (in the context of eqn. 21 the source function Jem, sc gathers both emission and scattering). Framing the RTE in terms of the optical path is less useful in the context of a spectral treatment, since then the relationship $d\tau/dz$ is non-unique (ie. varies with wavelength).

Plane-parallel atmosphere

As noted above, in some circumstances it may be appropriate to consider that the radiation intensity $I(\mathbf{x}, \mathbf{s})$ varies only with height z, thus $I = I(z, \mathbf{s})$.

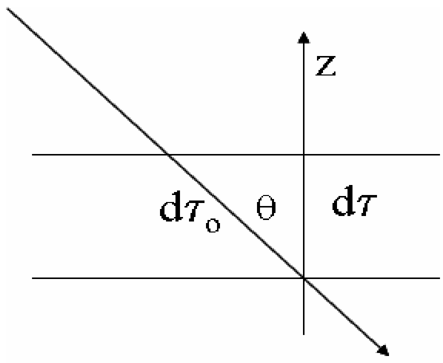

Figure 3: Replacing the increment of vertical distance with the projection onto the vertical axis of the increment in optical path: $d\tau = \; d\tau_0 \cos\theta$, facilitating use of $\tau = \tau(z)$ as the vertical coordinate. It is conventional to define $\tau = 0$ at the top of the atmosphere ($z = \infty,\ p = 0$). The value of τ at sea-level or ground depends on the waveband in question, and on the condition of the overlying atmosphere. Considering for the moment solar radiation in a clear sky, values at sea level are strongly wavelength-dependent, but in the range of about 1 - 0.1. By contrast, in the presence of a thick layer of cloud, τ may be as large as 50 or more.

Now the largest radiative flux convergences in the atmosphere will occur in cloudy or hazy atmospheres, and it is common to introduce the further simplification/assumption that the intensity $I(\tau, \mathbf{s})$ depends on the zenith angle, but not the azimuth; ie. that the dependence of I on direction $\mathbf{s}$, in general entailing dependence on two angles, reduces to a dependence on the emergent zenith angle, or μ: $I = I(\tau, \theta)$. For example, suppose I is the diffuse solar intensity. The simplification under discussion is valid if, when looking up at the sky at any angle relative to the zenith, we see the same intensity no matter which compass direction we face.

Radiation regimes

Let L be a characteristic dimension of the medium, and (as earlier) κ_λ the monochromatic (mass) absorption coefficient (it can be shown that the photon mean free path is $(\rho^{-1}\kappa_\lambda^{-1})$. The optical thickness of the medium is defined

$$\tau_{0\lambda} = \frac{L}{(\rho\,\kappa_\lambda)^{-1}} \tag{22}$$

Optically thick limit

Here $L \gg (\rho\kappa_\lambda)^{-1}$, and (if consider the process on some length scale very large compared to the mean free path) radiation transfer can be described as a diffusion process. Intuitively, we may think of the situation by an analogy with molecular collisions. Photons travel along a random distance (characterized by their mean free path) between incidents of being absorbed or scattered off atoms or molecules. According to André and Mahrt (1982; p871) Brunt (1934; *Physical & Dynamical Meteorology*) gave a theoretical based for the radiative diffusion formalism. Sparrow and Cess (p205) give this as follows: if $\vec{q}_\lambda$ is the monochromatic radiative energy flux density, then

$$\vec{q}_\lambda = - \frac{4}{3\,\rho\,\kappa_\lambda}\,\nabla\,e_\lambda \tag{23}$$

There is an interesting and useful implication. Bearing in mind that $\nabla\,e_\lambda \propto \nabla T$, the spatial gradient in the black body spectral emissive power e_λ, which drives the radiation energy flux, can arise only due to a temperature gradient. The mean free path for photons of a wavelength susceptible to absorption by CO_2 is very short compared to the depth of the atmosphere (in the prevailing circumstances, where bulk CO_2 concentration is about 370 ppm). Thus the diffusion regime prevails. If further CO_2 is added, the mean free path is decreased, reducing the effective diffusivity, so that a greater temperature gradient is needed to drive a given radiant energy flux. This vanquishes the (spurious) claim that further addition of greenhouse gases does not lead to an increase in near ground temperature. Furthermore since radiative heating rate (in band λ) is $-\nabla \bullet \vec{q}_\lambda$, this heating rate is determined by the *local* curvature of the temperature field. In Sec.() we see that more generally, radiative heating is determined by a *non-local* measure of the temperature curvature.

Optically thin limit

Here every element of the fluid interacts directly with bounding surfaces.

Case of a non-scattering but emitting medium

This (for example) is the case of longwave radiation ($\lambda \sim 4 - 100\ \mu$m) in an ideally particulate-free atmosphere (since λ is vastly greater than molecular diameter, and there are no other particles to interact with). The source function, by restriction entirely due to emission, is the Planck function.

As noted above the Planck function is isotropic (a black body radiates with equal intensity into every equal element of solid angle making up the hemisphere it faces). The Stefan-Boltzmann law for the emittance ([W m^{-2}]) of a black body is

$$\theta = \pi \int B_\lambda(T)\ d\lambda = \sigma T^4 \tag{24}$$

The factor π (rather than 2π, the solid angle subtended by a hemisphere) pops out from an integration of the projection factor ($\mathbf{s} \bullet \hat{k}$) over the 2π steradians of the hemisphere,

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \cos\theta\ \sin\theta\ d\phi\ d\theta = \pi \tag{25}$$

Now, returning to the RTE since the atmospheric gases are selective absorbers and emitters, a spectral description is essential, so we have to revert to

$$\mu\ \frac{dI_\lambda(\tau_\lambda, \theta)}{d\tau_\lambda} = -\ I_\lambda(\tau_\lambda, \theta) +\ B_\lambda(\tau_\lambda) \tag{26}$$

where now the increment $d\tau_\lambda \equiv \kappa_\lambda\, \rho\, dz$ involves the 'spectral mass extinction coefficient' (and we note extinction will be entirely due to absorption, so this will be the 'spectral mass absorption coefficient').

In the context of multi-spectral satellite remote sensing using an radiation bands in the atmospheric window, eqn. (26) is the basis for inversion to deduce the atmospheric temperature profile $T(z)$. We shall return to it later, in the context of a discussion of longwave radiative divergence in the nocturnal ASL.

Case where $J = 0$

Since the atmospheric gases emit a negligible amount of energy in the solar waveband, if we neglect multiple scattering (which first knocks a photon out of the path **s** of the beam, then kicks it back onto another path parallel to and thus indistinguishable from **s**) the RTE for a plane-parallel atmosphere reduces to

$$\mu \frac{dI(\tau, \phi, \theta)}{d\tau} = -\, I(\tau, \phi, \theta) \tag{27}$$

(here the assumption of azimuthal symmetry no longer applies) and we have the familiar exponential attenuation law.

Interactions of radiation and matter

Since extinction (loss of intensity) may be the result either of absorption (complete removal of photons) or of photon path-deflection (scattering), one splits the extinction coefficient as:

$$\beta = \kappa + \gamma \tag{28}$$

where κ is the absorption coefficient, and γ the scattering coefficient. Dividing by β,

$$1 = \frac{\kappa}{\beta} + \frac{\gamma}{\beta} \tag{29}$$

where $\omega_o \equiv \gamma/\beta$ is the scattered fraction of extinguished intensity, known as the "single scattering albedo", and $(1 - \omega_o) \equiv \kappa/\beta$ is the absorbed fraction.

Note that when one considers solar radiation in a real atmosphere composed of molecules and particulates (aerosols, cloud droplets, etc.), then "Since scattering by particles and absorption by gases occur simultaneously, the exact amount of absorber along the light path cannot be known" (Fouquart and Bonnel, 1980).

Scattering and absorption

Scattering is defined (Sparrow & Cess, 1978) as any change in the direction of propogation of photons, and can be due to local inhomogeneities in the medium, perhaps due to foreign bodies such as (case of the atmosphere) droplets or aerosols, or due to interaction with molecules of the medium. In 'coherent scattering' there is no change in frequency of the photon.

In general a scattering volume (finite volume of the continuum) will contain many scatterers; thus light scattered off one particle may be scattered again by another, and so on. Thus a photon that enters a scattering volume with direction **s** may potentially exit that volume with the same direction of propagation after an arbitrary number of scattering interactions... all that complexity has to be integrated into the extinction coefficient, and our RTE is a superficial (macroscopic) treatment.

Absorption of emission of radiation is associated within transitions in the energy levels of the atoms or molecules of the medium., and these transitions are classified as bound-bound, bound-free (interaction with the photon produces an electron and an ion) and free-free (interaction with a free electron). The first of these, involving quantized changes in the electronic, vibrational or rotational state, entails spectral lines; the latter two entail a spectral continuum (ie. the electron may have any energy).

Absorption and thermal emission in the atmosphere are considered isotropic processes (the one the reverse of the other) - ie. a unit mass of air has the same probability of absorbing a photon, regardless of the direction it came from; and thermal emission from unit mass is evenly distributed with respect to direction). However scattering preserves a memory of the direction of incidence, ie. is not isotropic, and so a formalism is required.

Scattering Function

Define the scattering function $p(\phi, \theta;\ \phi', \theta')$ such that $p(\phi, \theta;\ \phi', \theta')\ d\omega/4\pi$ is the probability that a photon incident at angle (ϕ', θ') will be scattered into solid angle $d\omega$ centred on emergent direction (ϕ, θ). A scattered photon (by definition of "scattering") must emerge at some angle: thus we have the normalization property of p,

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} p\,(\phi, \theta; \phi', \theta')\ \frac{\sin\theta\ d\theta\ d\phi}{4\pi} = 1 \tag{30}$$

(where $d\omega = \sin\theta \; d\theta \; d\phi$). For "isotropic scattering", by definition $p(\phi, \theta;\; \phi', \theta') \equiv 1$.

As an example, suppose we take $\phi' = \theta' = 0$ and assume that the scattering function has azimuthal symmetry: then the forward-scattered fraction would be

$$f = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} p(\theta) \; \frac{\sin\theta \; d\theta \; d\phi}{4\pi} \tag{31}$$

Evaluation of the scattering function is a problem in physics; various exact analytical solutions are available for the interaction of an incident plane electromagnetic wave with a single particle. Of course, in application to finite volumes of the atmosphere the scattering volume will contain many types of particles, so the bulk scattering function must represent the global consequence of the ensemble of scatterers on incident E/M waves of the given character (single wavelength λ or waveband $\lambda_1 \rightarrow \lambda_2$).

Rayleigh Scattering

Rayleigh scattering is the scattering of unpolarized light from very small particles (radius $r << \lambda$, the wavelength of the radiation). Thus the scattering of sunlight by the molecules of the atmospheric gases is Rayleigh scattering.

The scattering probability in this case is highly dependent on wavelength, varying as λ^{-4}. Thus the "Rayleigh optical depth" of the earth's atmosphere (clear air, absorption neglected, extinction entirely through scattering) is strongly dependent on the wavelength in question (see Table 1). In particular, the optical depth for blue light is much greater than for red light, implying much stronger scattering of blue than red light, out of the solar beam (and thus available to strike our eye as diffuse solar radiation, ie. as a component of the sky colour).

For Rayleigh scattering of incident unpolarized radiation, the scattering function (here written s.t. emergent angles are measured w.r.t. the incident direction) has azimuthal symmetry, and is given by[6]:

$$p(\theta) = \frac{3}{4} \left(1 + \cos^2 \theta\right) \tag{32}$$

The scattering function gives the distribution of the angle of scattering, for photons that *are* scattered - it does *not* give the probability of scattering. This is a bit confusing; what is the difference between a photon's not being scattered, and its being forward-scattered onto exactly the same path as it had before the scattering interaction?

In any case, it is interesting to note that the Rayleigh scattering function is not wavelength-dependent, and so exerts no influence on the red/blue character of the atmosphere... it is (rather) the wavelength dependence of the probability of scattering that controls sky colour.

Exercises

1. Show from eqn (31) that the forward integral of the Rayleigh scattering function is $f = 1/2$

2. Show that the Rayleigh scattering function has maxima $p = 3/2$ at $\theta = 0^o, 180^o$ and minima $p = 3/4$ at $\theta = 90^o, 270^o$

[6] Liou calls this $P(\cos\phi)$, ie. his scattering deflection angle ϕ is our θ.

Table 1: Rayleigh optical thickness of earth's atmosphere at sea level, under clear skies. (Pertains to extinction of a beam from a source at the zenith).)

λ [μm]	τ^R
0.25	2.74
0.30	1.25
0.35	0.650
0.40	0.373
0.45 ("blue")	0.229
0.50	0.149
0.55	0.101
0.60	0.0708
0.65	0.0512
0.70 ("red")	0.0379
0.75	0.0287
0.80	0.0221
0.85	0.0173
0.90	0.0138
0.95	0.0111
1.00	0.0090

Mie Scattering

The atmosphere always contains dust, haze, and other types of particles (cloud and rain drops) in addition to the molecular size particles of the atmospheric gases, and their radiative effects must be considered. Scattering by spherical particles of radius r about equal to or greater than the wavelength λ is given by the Mie theory, which however is too complicated to be discussed here.

Two-stream model for diffuse solar radiation

We have seen that at the most fundamental level, radiative transfer is quantified by the intensity I. We have seen, too, how the heating rate due to radiative divergence is calculated from I. Provided we can supply in sufficient detail the material properties κ (etc.) and realistically model scattering (of photons of any wavelength) by any and all forms of material in the atmosphere, there is no fundamental barrier to arbitrarily accurate computations of radiation distribution in the atmosphere. Unfortunately such a rigorous and general treatment is beyond available computing resources in the context of global climate modelling, and may well always be; and furthermore specifying the distribution of particulate matter and the scattering and absorption coefficients is no small task. Simplified models are a necessity, and Stephens (1992) considers that in the context of parameterizing the interaction of radiation and cloud it is important to distinguish

- Inherent optical properties; functions of cloud microphysics that may not be able to be supplied by large scale models

- Apparent optical properties; properties that are apparent o the medium and come about through illumination of the cloud by radiation

It is common to resort to some variant of the following "two-stream" (upward, downward) model[7], and we will here visualize that model in terms of its application to the profile of diffuse solar radiation (formulae of the same nature apply for longwave radiation).

We begin by defining $F^+(\tau)$ as the upward and $F^-(\tau)$ the downward stream of radiation; these are both flux densities, having units [W m^{-2}], and the net diffuse solar radiative energy flux density (defined positive for net energy transport towards earth) is

$$F^* = F^- \; - \; F^+ \tag{33}$$

Now, for either component of the flux we may describe the change over optical path increment $d\tau$ due to *extinction* (with effect of scattering to be

[7]According to Lenoble (1985; Radiative Transfer in Scattering and Absorbing Atmospheres: Standard Computational Procedures), this model *cannot* be derived rigorously from the RTE. Thus the two-stream model must be regarded as heuristic, though presumably it can be 'calibrated' by adjusting the parameter inputs (viz., single scattering albedo; forward and backward scattering probabilitities) in reference to measurements or to a fully-directional model. Stephens (1992), noting that all current radiation schemes for large scale models are based on the "plane parallel version of the radiative transfer equation" (ie. two-stream approximation) states that such a procedure "may produce arbitrarily large errors due to the subgrid scale effects of cloud morphology" (ie. lateral radiative exchange due to cloud boundaries). Be that as it may, many authors adopt this description as a starting point in the analysis of atmospheric radiation. Liou (2002) gives the two stream model in at least two forms; that of his eqns (6.5.8), and in the form of his generalized two-stream approximation, eqns(6.5.27).

accounted also, below) as

$$dF = -F\ d\tau \equiv\ -F\ (\kappa/\beta + \gamma/\beta)\ d\tau \tag{34}$$

Let us (as earlier) define $\gamma/\beta = \omega_o$, the "single-scattering albedo," ie. the scattered fraction of the extinguished radiation; then $1 - \omega_o$ is the absorbed fraction of extinction. Since we now admit of only two directions of transfer, our description of the directionality of scattering reduces to the specification of

- b, the backscattered fraction of the diffuse radiation
- b_o, the up-scattered fraction of the scattered part of the solar beam
- f_o $(= 1 - b_o)$, the down-scattered fraction of (the scattering from) the solar beam

and we require empirical data or a suitable physical model of the appropriate scattering process to evaluate these parameters. (Note that the form of the equations to follow indicates that the net effect of *multiple* scattering is handled by means of these adjustable parameters.)

Let $S(\tau)$ [W m^{-2}] be the beam strength measured on a horizontal surface: then $S(0) = 1370\mu_o$ where $\mu_o = \cos(\theta_o)$, θ_o being the solar zenith angle. Now, over a positive (downward) increment $d\tau$, the change in F^- is made up of loss by absorption and scattering; gain by forward scattering from itself; gain by backward scattering of F^+; and gain by downward scattering of the beam.

Thus

$$dF^- = \ -d\tau\ F^- + \ (1-b)\ \omega_o\ d\tau F^- + \ b\ \omega_o\ d\tau\ F^+ + \ (1-b_o)\ \omega_o\ d\tau\ \frac{S(\tau)}{\mu_o}$$

and a similar result holds for the change in F^+. Rearranging[8]

$$\begin{aligned}
\mu_1\,\frac{dF^-}{d\tau} &= F^+\ (\omega_o\ b) - \ F^-\ (1-\omega_o+b\omega_o) + \ \omega_o(1-b_o)\ \frac{S(\tau)}{\mu_o} \\
\mu_1\frac{dF^+}{d\tau} &= F^+\ (1-\omega_o+b\omega_o) - \ F^-\ (\omega_o b) - \ \omega_o b_o\ \frac{S(\tau)}{\mu_o} \qquad (35)
\end{aligned}$$

where

$$\begin{aligned}
\frac{dS}{d\tau} &= -\ \frac{S}{\mu_o} \\
S(\tau) &= S(0)\ \exp\left(-\ \frac{\tau}{\mu_o}\right) \qquad (36)
\end{aligned}$$

The boundary conditions are:

$$\begin{aligned}
S(0) &= 1370\mu_o \\
F^-(0) &= 0 \\
F^+(\tau_{gnd}) &= \alpha\left(F^-(\tau_{gnd}) + S(\tau_{gnd})\mu_o\right) \qquad (37)
\end{aligned}$$

where α is the albedo of the ground (the second of these equations states that there is no incoming diffuse radiation at the top of the atmosphere). Analytical solution is probably possible, and numerical solution is easy. The

[8]The projection factor μ_1 "allows (approximately) for the mean obliquity of the rays to the vertical direction" (Paltridge and Platt, "Radiative processes in Meteorology and Climatology", p74) but is commonly suppressed with corresponding readjustment of the coefficients of all other terms (eg. Liou, p108). Schuster (1905; "Radiation through a foggy atmosphere", Astrophys. J., Vol. 21, 1-22) is credited with first use of a two-stream formulation.

link between increments in τ and the corresponding increments in pressure (or decrements in height) is straightforward,

$$\frac{d\tau}{dp} = -\frac{d\tau}{dz}\left(-\frac{\partial p}{\partial z}\right)^{-1} = \frac{-1}{\rho g}\frac{d\tau}{dz} \tag{38}$$

Now if β is the mass extinction coefficient, $d\tau = \beta\, \rho\, dz$, thus

$$\frac{d\tau}{dp} = -\frac{\beta}{g} \tag{39}$$

To progress to specific predictions of diffuse radiation in (say) cloudy skies, we need specifically to know the profile of the extinction coefficient, and the other parameters that have been introduced. This connects with cloud microphysics.

Long wave radiation in the pure, clear atmosphere

It isn't hard to see that large scale meteorology has no choice but to deal with heating/cooling by radiative flux divergence, due to the presence of clouds. But we don't tend to get clouds in the surface layer (exception: fog) and so in micro-meteorology, the tendency has been to ignore radiative divergence, so that radiative fluxes are considered only as they enter the surface energy budget, and thus bear on the magnitude of the injected (or ejected) surface fluxes of heat and water vapour.

The role of longwave radiative flux divergence in the cooling of the nocturnal atmospheric surface layer (ASL) is neglected in the simplest explanatory paradigm (eg. Delage, 1974; Nieuwstadt, 1980) wherein cooling of the ASL is driven exclusively by divergence $\partial Q_H/\partial z$ of the turbulent convective heat flux $Q_H = \rho c_p \overline{w'\phi'}$, that flux being negative (ie. heat transport from air to ground). This is known to be wrong, however. From a study that coupled measurements and spectrally-resolved radiative calculations so as to evaluate the terms in the conservation equations for heat and water vapour, Schaller (1977) observed that "during the clear night radiative cooling exceeds the cooling caused by the sensible heat flux." André et al. (1978) modelled the nocturnal evolution of the Wangara experiment, and likewise concluded that longwave divergence is "more important than turbulent transport $T = -\partial \overline{w'\phi'}/\partial z$ except close to the ground" such that "nocturnal mean structure is... driven principally by radiative transfer." André and Mahrt (1982) state that "The crucial role of clear-air radiative cooling appears to preclude successful relation of the inversion depth to surface fluxes, as has been frequently attempted in the literature."

First, let's note that if our focus is radiative heating, then since it is the divergence of the radiative flux that is of interest we need only account for the spectral bands of the vertical radiative flux densities that *interact with the atmospheric gases...* parts of the spectrum for which air is transparent need not be accounted for (this is obvious, but easy to lose sight of as many authors regard it as too obvious to be stated).

Following on from Sec.(), let's address long wave radiation by assuming only the molecules of air participate. We need only be concerned with absorption and emission (ie. no scattering), but these exchanges are "very spectrally dependent" (Paltridge and Platt, 1976; hereafter P&P). The absorption occurs in spectral bands, each composed of many individual lines where absorption corresponds to changes in molecular vibrational or rotational energies. "At wavelengths in the centre of the molecular bands, the infrared radiation can be absorbed within a few metres of atmosphere" (P&P), and we could interpret this in terms of a mean free path for photon interactions with the atmospheric gases.

In relation to the longwave region of the spectrum ($4 - 100\,\mu$m), the most important gases are H_2O, CO_2, O_3 and CH_4. Carbon dioxide is fairly well-mixed in the troposphere and stratosphere, the diurnal and seasonal fluctuations (due to plant photosynthesis) being small relative to the mean of order 370 ppm. The most important absorption band for CO_2 lies at $14.7\,\mu$m (vibration-rotation), and others lie at 4.3, $2.7\,\mu$m.

Relative variability in water vapour concentration is far stronger, and water vapour is an important greenhouse gas in the in the lower troposphere where humidity can be high. The important water vapour bands are at $6.3\,\mu$m (vibration-rotation) and $20\,\mu$m (pure rotation). There is also a 'continuum band' covering $8 - 14\mu$m.

The relevant spectral RTE is eqn. (20), in which we drop the scattering term and replace the extinction coefficient with the absorption coefficient, viz.

$$\mu \, \frac{dI_\lambda^+(z,\mathbf{s})}{dz} = - \, \kappa_\lambda \, \rho \, I_\lambda^+(z,\mathbf{s}) + \; \kappa_\lambda \, \rho \, B_\lambda(z) \tag{40}$$

(c.f. Paltridge and Platt, 1976, eqn 4.2) or

$$\frac{dI_\lambda^+(z,\mathbf{s})}{dz} + \frac{\kappa_\lambda\,\rho}{\mu}\,I_\lambda^+ = \frac{\kappa_\lambda\,\rho}{\mu}\,B_\lambda(z) \tag{41}$$

Note (see P&P, p71) that we are making two assumptions: a horizontally plane parallel atmosphere, and, that radiation emission is isotropic. As P&P note, "taken together, they allow an effective 'two-stream' approximation whereby the long-wave field can be specified in terms only of the upward and downward components of the flux density $F\uparrow$ and $F\downarrow$ respectively."

Skipping the details, one may (rigorously) derive that

$$\begin{aligned} F_\lambda^+(z) &= e_\lambda(z) - \int_0^z \widetilde{T}_\lambda(z,z')\,\frac{\partial e_\lambda(z')}{\partial z'}\,dz' \\ &+ \widetilde{T}_\lambda(z,0)\,\left[e_\lambda(T_{gnd}) - e_\lambda(T_{air,gnd})\right] \end{aligned} \tag{42}$$

(Rodgers and Walshaw, 1966; eqn 2) where the second line on the r.h.s. is negligible unless the air at $z=0$ is at a different temperature from the ground surface. In eqn.(42) the flux transmittance

$$\widetilde{T}_\lambda(z_2,z_1) = 2\,Ei_3\left(\int_{z_1}^{z_2} k_\lambda(z')\,\rho(z')\,dz'\right) \tag{43}$$

where

$$Ei_3\,(\alpha) = \int_1^\infty e^{-\alpha x}\,x^{-3}\,dx \tag{44}$$

Note that

$$\widetilde{T}_\lambda(z,z) \equiv 1 \tag{45}$$

which follows from

$$Ei_3\,(0) = \int_1^\infty x^{-3}\,dx = \; 1/2 \tag{46}$$

Clearly one ought to be cautious and systematic about the ordering of the two parameters in $\widetilde{T}_\lambda(z_2, z_1)$, and I have taken the convention that $z_2 \geq z_1$. However authors (eg. Rodgers and Walshaw, 1966; eqn 2) are *not* systematic in this regard. Provided one takes the interpretation that $Ei_3()$ is always to have a positive argument (that entails an integration of $\kappa_\lambda(z')\rho(z')dz'$ between the two arguments) no harm is done. However wherever the derivative of $\widetilde{T}_\lambda(z, z')$ is wanted, note that

$$\frac{\partial \widetilde{T}_\lambda(z,z')}{\partial z} = \begin{cases} > 0 & z' < z \\ < 0 & z' > z \end{cases} \tag{47}$$

or simply

$$\frac{\partial \widetilde{T}_\lambda(z,z')}{\partial z} = -\,\frac{\partial \widetilde{T}_\lambda(z',z)}{\partial z} \tag{48}$$

Now from the equations for F_λ^+, F_λ^-, one can show that the heating rate is

$$\begin{aligned} -\frac{\partial}{\partial z}\left(F_\lambda^+ - F_\lambda^-\right) \;=\; & \int_0^z \frac{\partial \widetilde{T}_\lambda(z,z')}{\partial z}\,\frac{\partial e_\lambda(z')}{\partial z'}\,dz' + \int_z^{Z_T} \frac{\partial \widetilde{T}_\lambda(z',z)}{\partial z}\,\frac{\partial e_\lambda(z')}{\partial z'}\,dz' \\ & - \frac{\partial \widetilde{T}_\lambda(z,0)}{\partial z}\,\left[e_\lambda(T_{air,gnd}) - \; e_\lambda(T_{gnd})\right] \\ & + \frac{\partial \widetilde{T}_\lambda(z_T,z)}{\partial z}\,\left[e_\lambda(T_{air,top}) - e_\lambda(z_T)\right] \end{aligned} \tag{49}$$

The boundary terms (ground, upper boundary of computational domain) should be insignificant provided the distance from z to the boundary is several times the photon mean free path $\rho^{-1}\kappa_\lambda^{-1}$ and indeed may well vanish entirely (e.g. z_T placed at upper limit of atmosphere, and ground a full spectrum black body emitter at same temperature as the air in contact with it).

Interpretation: Recall that $\partial_z e_\lambda \propto \partial_z T$ and so eqn.(49) claims that, discounting for now the effect of boundary terms, the longwave radiative heating rate at height z is determined from a weighted height average of the temperature gradient at all heights $0 \leq z' \leq z_T$. Furthermore, bearing in mind that $\widetilde{T}_\lambda(z_2, z_1) \leq 1$ gets smaller as the interval $z_2 - z_1$ is increased (and vice versa) *the weighting function is positive for levels z' above z, and negative for heights below z.* Fig.(4) shows a sequence of simplified temperature profiles (not all of which are realistic), with constant (but differing) temperature gradient above and below the level (z) in question. The change in sign of the weighting function means that it is the *difference* in the temperature gradient that determines the heating rate, and this evidently can be considered to be the *curvature* $\partial_{zz}T$.

Afterword: The measurements of Funk (1960) emphasize that in general, the neglect of aerosols (in particular, haze and fog) is a serious shortcoming of the present treatment of longwave radiative divergence.

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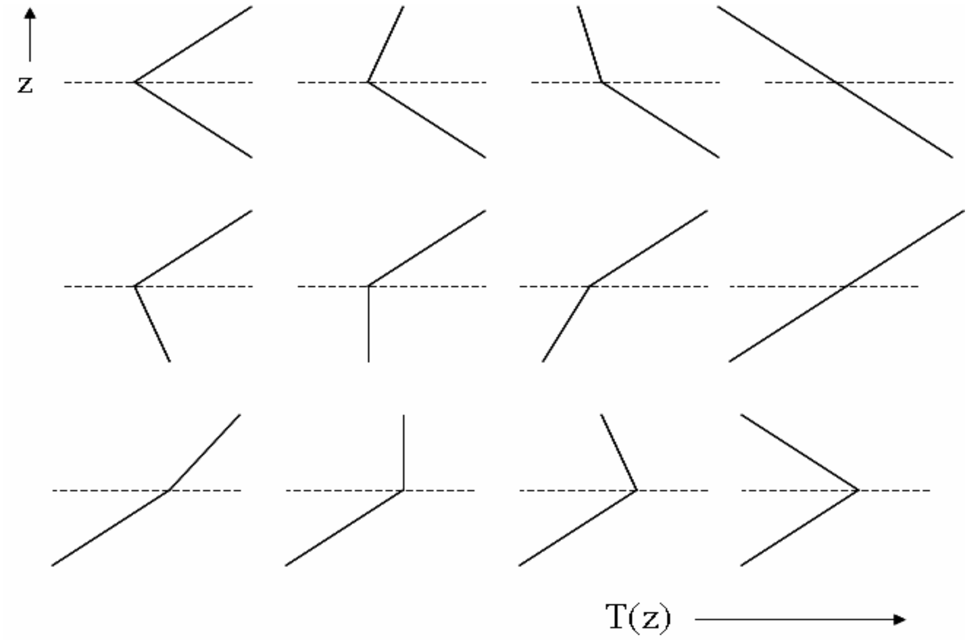

Figure 4: Simplified temperature profiles $T(z')$, in relation to the rate of heating due to longwave radiative flux divergence at a fixed height z. Heating rate depends on the *difference* in slope $(\partial T/\partial z)^{z'>z} - (\partial T/\partial z)^{z'<z}$, which is related to the curvature $\partial^2 T/\partial z^2$ of the temperature profile. Eqn.(49) indicates the rate of heating is determined non-locally, ie. by the difference between weighted height-integrals of the temperature-gradient above and below the observation level. The top two rows correspond to radiative heating (except the right-most profiles where heating vanishes) $\partial^2 T/\partial z^2 \geq 0$, while the lowest row corresponds to radiative cooling ($\partial^2 T/\partial z^2 < 0$).

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