EAS471, "Atmospheric Modelling" Exam 23 April, 2010

Professor: J.D. Wilson   Time available: 120 mins   Value: 35%

Please answer in the exam booklet. Symbols have their usual meteorological interpretation. Some data are given at the back.

Multi-choice $(22 \text{ x } \frac{1}{2}\% = 11\%)$

1. If the vector $\vec{F}$ represents the convective flux density of a certain scalar property ϕ and $\vec{u}$ the velocity field, then ____
   (a) $\vec{F} = \vec{u} \cdot \nabla\phi$
   (b) $\vec{F} \cdot \vec{u} = 0$
   (c) $\nabla \cdot \left(\vec{F}\,\vec{u}\right) = 0$
   (d) $\vec{F}$ and $\vec{u}$ are perpendicular
   (e) $\vec{F} = \vec{u}\,\phi$

2. The units of a diffusivity (as appears, for example, in $\partial\phi/\partial t = K\,\partial^2\phi/\partial x^2$) are ____
   (a) $\mathrm{m\,s^{-1}}$
   (b) $\mathrm{m\,s^{-2}}$
   (c) $\mathrm{m^2\,s^{-1}}$
   (d) $\mathrm{m^2\,s^{-2}}$
   (e) $\mathrm{kg\,m\,s^{-2}}$

3. A continuous random variable q, defined on the range $-\infty \le q \le \infty$, belongs to a probability distribution whose probability density function is $f(q)$. If

$$\overline{q} \equiv \int_{-\infty}^{\infty} q\, f(q)\, dq \ = 0$$

   then the variance of q is given by ____
   (a) $\int_{-\infty}^{\infty} q^2\, f(q)\, dq$
   (b) $\int_{-\infty}^{\infty} f(q)\, dq \equiv 1$
   (c) $\int_{-\infty}^{\infty} f(q)\, dq$
   (d) $\int_{-\infty}^{\infty} dq$
   (e) ∞

4. In the context of numerical solution of the advection equation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = 0$$

using finite differences in time (Δt) and space (Δx), a stability criterion on the Courant number arises. This criterion involves a ratio of velocities, one of which, namely ____ , is ____ in origin

   (a) U; numerical
   (b) $\Delta x / \Delta t$; numerical
   (c) $\Delta x / \Delta t$; physical
   (d) $\Delta t / \Delta x$; physical
   (e) $\Delta \phi / \Delta x$; numerical

5. The Navier-Stokes equation (expressing conservation of momentum) and the conservation equation for a passive, non-reacting ("tracer") species are given as data, as are conventions for representing the velocity vector. The advection term $u_j \; \partial c / \partial x_j$ in the tracer conservation equation may alternatively be written ____

   (a) $\vec{u} \cdot \nabla c$
   (b) $\nabla \, (c \, \vec{u})$
   (c) $u \, \frac{\partial c}{\partial x}$
   (d) $u \, \frac{\partial c}{\partial x} + v \, \frac{\partial c}{\partial y} + w \, \frac{\partial c}{\partial z}$
   (e) both a and d are valid

6. Again referring (if necessary) to the Navier-Stokes equation and the tracer conservation equation (given as data, along with definitions of Reynolds, Rossby & Peclet numbers), the Lagrangian approach to treating dispersion of a tracer is valid in the limit of ____

   (a) infinitely small Reynolds number
   (b) infinitely large Reynolds number
   (c) infinitely small Peclet number
   (d) infinitely large Peclet number
   (e) infinitely large Rossby number

7. According to the prescription covered in EAS471 (i.e. following Molinari, 1993), a "mesoscale" model of the atmosphere is ____ , and has horizontal gridlengths in the range ____

   (a) non-hydrostatic; $10 \leq \Delta \leq 50$ km
   (b) hydrostatic; $10 \leq \Delta \leq 50$ km
   (c) non-hydrostatic; $1 \leq \Delta \leq 10$ km
   (d) hydrostatic; $1 \leq \Delta \leq 10$ km
   (e) Eulerian; $\Delta \leq 10$ m

8. Let $\delta T(z)$, $\delta q(z)$ be adjustments to resolved temperature and specific humidity made in the layer $z_B \le z \le z_T$ of a particular grid column upon application of a cloud scheme(s) of the physics package of an NWP model. If these adjustments satisfy

$$\begin{aligned} -\,L\,\rho\,\delta q(z) &= \rho\,c_p\,\delta T(z), \quad (\delta T \ge 0)\ , \\ q(z) + \delta q(z) &= q_*(T + \delta T, p)\ , \end{aligned}$$

where $q_*(T)$ is the saturation specific humidity for temperature T, then the package that was activated corrects for ____

(a) moist convection

(b) dry convection

(c) non-convective resolved-scale supersaturation (assumed to result in condensation)

9. The two diagnostic conditions pertaining to the resolved state of the atmosphere and that cause the activation of the "Kuo scheme" for unresolved deep convection are ____ (where ∇_H is the horizontal grad operator, and $\vec{v}_H$ the resolved horizontal velocity)

(a) low level convergence $\nabla_H \cdot \vec{v}_H < 0$, and conditional instability of a deep layer

(b) low level convergence $\nabla_H \cdot \vec{v}_H > 0$, and absolute instability of a deep layer

(c) low level convergence $\nabla_H \cdot \vec{v}_H < 0$, and absolute instability of a deep layer

(d) existence of CAPE (convectively available potential energy) exceeding 1000 $\mathrm{J\,kg^{-1}}$, and supersaturation of a deep layer

10. The Kuo scheme, when activated, computes the quantity

$$M_t = -\int_0^\infty \nabla_H \cdot (\rho\, q\, \vec{v}_H)\ dz + E_0$$

where E_0 is the surface evaporation rate, q is resolved specific humidity, and other quantities are as above. This quantity represents ____

(a) a supply of latent heat (and, ipso facto, water vapour) available to build convective clouds

(b) the amount of precipitation produced by the scheme

(c) the energy used to raise the environmental temperature (i.e. resolved model state $\widetilde{T}(z)$, pre-correction) to the temperature $T_c(z)$ of the diagnosed cloud

(d) local cloud fraction, i.e. fraction of sky occupied by deep convective cloud

11. The shortest wave that can be represented on a grid with spacing Δx has wavelength

(a) $\pi/\Delta x$

(b) $2\pi/\Delta x$

(c) $\Delta x/2$

(d) Δx

(e) $2\Delta x$

12. The Canadian Meteorological Centre's Global Environmental Multiscale (GEM) model for Numerical Weather Prediction treats (resolved) advection terms by a "semi-Lagrangian" method. A fully Lagrangian approach to solving the momentum equations is not possible because ____

    (a) velocity (and momentum) are conserved properties
    (b) velocity (and momentum) *are not* conserved properties
    (c) momentum is absorbed by the surface underlying the atmosphere
    (d) fluid elements are not absorbed by the surface
    (e) too many grid cells would be required

13. Suppose the distribution of a property $\phi = \phi(\mathbf{x}, t)$ in a fluid/gas is governed by

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F_i}{\partial x_i} + Q(\mathbf{x}, t) ,$$

    where $\mathbf{x} \equiv x_i$ signifies position (note: you might compare this "ϕ-eqn" with the species conservation equation given as data). The transport term in this equation is ____

    (a) $\partial \phi / \partial t$
    (b) Q
    (c) F_i
    (d) $-\partial F_i / \partial x_i$
    (e) $\int_{-\infty}^{t} \phi(\mathbf{x}, t)\, dt$

14. Suppose the distribution of a property $\phi = \phi(\mathbf{x}, t)$ in a fluid/gas is governed by

$$\frac{\partial \phi}{\partial t} = -\frac{\partial F_i}{\partial x_i} + Q(\mathbf{x}, t) ,$$

    (the "ϕ-eqn") where $\mathbf{x} \equiv x_i$ signifies position. This is a ____ general statement than the species conservation equation given as data, in the sense that ____

    (a) more; the flux density F_i of the above ϕ-eqn is generic, embracing any or all transport mechanisms (i.e. potentially radiation, convection, diffusion/conduction), and a volumetric source term is included, allowing for in situ production or destruction of ϕ
    (b) less; a volumetric source term is here included, allowing for in situ production or destruction of ϕ
    (c) less; no diffusion term appears in this equation for ϕ
    (d) less; no advection term appears in the ϕ equation

15. If ρ, T, c_p are the density, temperature and specific heat of air and $\vec{u}$ is the velocity field, the convective flux density of sensible heat is ____

(a) $\rho\, c_p\, \vec{u} \cdot T$
(b) $\rho\, \vec{u}\, T$
(c) $c_p\, \vec{u}\, T$
(d) $\rho\, c_p\, \vec{u}\, T$
(e) $\rho\, c_p\, \nabla \cdot (\vec{u}\, T)$

16. Suppose the profile of $f(x)$ along the x-axis is represented at discrete nodes, separated by interval Δx, and labelled J. If the derivative df/dx is represented as

$$\left(\frac{df}{dx}\right)_J = \frac{f_J - f_{J-1}}{\Delta x}$$

then the truncation error is of order

(a) Δx
(b) Δx^2
(c) $\sqrt{\Delta x}$
(d) $\left(x_{J+1} + x_{J-1}\right)/2$
(e) $f_J - f_{J-1}$

17. Suppose $f(x,t)$ defined on $-1 \leq x \leq 1$, $0 \leq t$ is governed by the heat equation

$$\frac{\partial f}{\partial t} = K\,\frac{\partial^2 f}{\partial x^2}$$

with boundary conditions $f(-1,t) = f(1,t) = 0$, and that $f(x,0) = \cos k\pi x/2$ where k is an arbitrary positive integer. Initially f has a maximum magnitude of 1, i.e. $|f|_{mx} = 1$. At later times the *true* state must satisfy

(a) $|f|_{mx} = k$
(b) $|f|_{mx} \geq 1$
(c) $|f|_{mx} \leq 1$
(d) $|f|_{mx} = \frac{K\,\Delta t}{\Delta x^2}$
(e) $|f|_{mx} = \frac{k\,\Delta t}{\Delta x^2}$

18. Gridpoint computations for the influence of unresolved scales of motion in the ABL on the resolved absolute humidity $\overline{\rho}_v$ will involve the equation

$$\left(\frac{\partial \overline{\rho}_v}{\partial t}\right)_{\text{physics}} = -\ [.]\ \overline{w'\rho_v'}$$

The missing operator "[.]" is ____

(a) $U\,\partial/\partial x + V\,\partial/\partial y$ (U, V the resolved horizontal velocity components)

(b) $W\,\partial/\partial z$ (W the resolved vertical velocity)

(c) $\partial/\partial x$

(d) $\partial/\partial z$

(e) $K\,\partial/\partial z$ (K the eddy diffusivity)

19. The "curvature" of a scalar field $\phi(x, y, z)$ is given by (or measured by) application of the ____ operator

(a) ∇

(b) $\nabla \cdot \nabla$

(c) $\hat{k} \cdot \nabla$

(d) $\hat{k}\ \times$

(e) $\hat{k}\ \times \nabla$

20. According to the Lax Equivalence Theorem, "If a difference equation is *consistent* with the differential equation it represents, then stability is the necessary and sufficient condition for convergence." Here the technical meaning of "consistent" is that

(a) truncation error must vanish in the limit of vanishing grid interval(s)

(b) truncation error must vanish in the limit of infinite grid interval(s)

(c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s)

(d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

21. According to the Lax Equivalence Theorem, "If a difference equation is consistent with the differential equation it represents, then stability is the necessary and sufficient condition for *convergence*." Here the technical meaning of "convergence" is that

(a) truncation error must vanish in the limit of vanishing grid interval(s)

(b) truncation error must vanish in the limit of infinite grid interval(s)

(c) the numerical solution ϕ^{num} equals the true (but generally unknown) solution ϕ to the differential equation in the limit of vanishing grid interval(s)

(d) the difference between the numerical solution ϕ^{num} to the difference equation and the (generally unknown) exact solution ϕ^* to the difference equation vanishes in the limit of vanishing grid interval(s)

22. The Random Displacement Model (RDM) for the vertical motion of a fluid element (or "particle") is

$$dZ = \frac{\partial K}{\partial z}\, dt + \sqrt{2\, K\, dt}\; r \; ,$$

where dZ denotes the increment in height during the time step dt, K is the eddy diffusivity, and r is chosen from a Normal distribution with zero mean and unit variance. The RDM

(a) is an Eulerian method

(b) requires the imposition of a spatial grid in order to allow computation of trajectories

(c) is a valid description of dispersion even in the near field of a source (i.e. for travel times t much smaller than the Lagrangian time scale for the vertical velocity)

(d) is a grid free (Lagrangian) treatment of turbulent convection that is *equivalent* to the "diffusion" model

(e) is also known as the "generalized Langevin equation" or the "first-order Lagrangian stochastic model"

Short answer: 3 x 8% = 24%

Answer any **three** questions from this section.

1. Perform a dimensional analysis to find the form of the law for the drag force F on a sphere of density ρ and radius R that is falling at velocity V through still air whose density and kinematic viscosity are respectively ρ_a, ν (the units of kinematic viscosity are $\mathrm{m^2\, s^{-1}}$).

2. Determine the 4×4 tridiagonal coefficient matrix $\mathbf{M}$ and the right hand side $\mathbf{B}$ in a matrix expression of form $\mathbf{M}\, \mathbf{\Theta} = \mathbf{B}$ for the numerical solution of

$$\frac{\partial^2 \theta}{\partial z^2} = 2\theta$$

on $0 \le z \le 1$, subject to $\theta(0) = 0$, $\theta(1) = 1$. Set up your solution with four, equi-spaced gridpoints indexed $J = (1, 2, 3, 4)$ positioned at $z_J = (0, 1/3, 2/3, 1)$. At internal gridpoints (J=2,3) the curvature is to be represented as

$$\frac{\partial^2 \theta}{\partial z^2} = \frac{\theta_{J+1} + \theta_{J-1} - 2\,\theta_J}{\Delta z^2} \; .$$

Note: you are not being asked to invert $\mathbf{M}$, nor to obtain the (numerical) solution vector $\mathbf{\Theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$.

3. Briefly summarize the Canadian Meteorological Centre's Global Environmental Multiscale (GEM) model of the atmosphere, as used for short range (48 hour) Numerical Weather Prediction. Your response should cover salient points in regard both to model dynamics and model physics (grid point computations).

4. In the context of grid point computations, which treat the atmospheric boundary layer as if it were horizontally-homogeneous, the conservation equation for the kinetic energy $k \equiv \frac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$ residing in the unresolved scales of motion can be approximated as:

$$\frac{\partial k}{\partial t} = K \left[\left(\frac{\partial \overline{u}}{\partial z} \right)^2 + \left(\frac{\partial \overline{v}}{\partial z} \right)^2 \right] - \frac{g}{\theta_0} \frac{K}{P_r} \frac{\partial \overline{\theta}}{\partial z} - \epsilon + \frac{\partial}{\partial z} \left(K_k \frac{\partial k}{\partial z} \right) ,$$

where θ_0 is the mean potential temperature of the layer, $(\overline{u}, \overline{v})$ and $\overline{\theta}$ are the resolved components of the horizontal wind and the potential temperature, K is the eddy viscosity, $K_h = K/P_r$ is the eddy diffusivity for heat (P_r is the turbulent Prandtl number), and K_k is the effective eddy diffusivity for the vertical diffusion of k.

Classify each term in this equation, and identify its conventional name — e.g. what term(s) represent "shear production"? Making reference to the flux Richardson number

$$R_i^f = \frac{g}{\theta_0} \frac{K/P_r}{K} \frac{\partial \overline{\theta}/\partial z}{(\partial \overline{u}/\partial z)^2 + (\partial \overline{v}/\partial z)^2} ,$$

explain the influence of atmospheric stratification on the energy of the unresolved motion (you may assume $P_r = 1$).

5. Assuming a hydrostatic atmosphere, the (vertical) vorticity equation is

$$\begin{aligned} \frac{\partial \zeta}{\partial t} &+ \vec{v}_H \cdot \nabla_H (\zeta + f) + \omega \frac{\partial \zeta}{\partial p} \\ &= - (\zeta + f) \nabla_H \cdot \vec{v}_H + \hat{k} \cdot \left(\frac{\partial \vec{v}_H}{\partial p} \times \nabla_H \omega \right) \end{aligned} \qquad (1)$$

where ζ is the relative vorticity, $\vec{v}_H$ is the horizontal wind vector and ω [Pa s^{-1}] the vertical velocity. Explain the further limitations/assumptions/substitutions and/or simplifications that lead to the quasi-geostrophic vorticity equation

$$\frac{d_g \zeta}{dt} \equiv \frac{\partial \zeta}{\partial t} + U_g \frac{\partial \zeta}{\partial x} + V_g \frac{\partial \zeta}{\partial y} + V_g \beta = f_0 \frac{\partial \omega}{\partial p}$$

where $\vec{V}_g$ is the Geostrophic wind vector defined at a reference latitude ϕ_0, f_0 is the Coriolis parameter at that latitude, and $\beta = (\partial f / \partial y)_{\phi_0}$ its northerly gradient.

Data

- $(\hat{i}, \hat{j}, \hat{k})$ are unit vectors along the Cartesian coordinate directions (x, y, z), and (u, v, w) are the corresponding Cartesian velocity components. Alternative notations for the velocity vector are $\mathbf{u}$, $\vec{u}$ and u_i, where the dummy subscript occurring in the last of these (and which could as easily have been written j or k or indeed any other symbol) takes on values of 1,2 or 3 corresponding to the three spatial axes.

- $$\frac{d\mathbf{u}}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{-1}{\rho} \nabla p + \mathbf{g} - 2\boldsymbol{\Omega} \times \mathbf{u} + \nu \nabla^2 \mathbf{u}$$

  The Navier-Stokes equation, expressing conservation of momentum of a Newtonian fluid: ρ the fluid density, p the pressure, $\mathbf{g}$ the gravitational acceleration vector, $\boldsymbol{\Omega}$ the angular velocity of the coordinate frame (occurring in the Coriolis term), ν the molecular kinematic viscosity. If V, L are velocity and length scales for the motion, a Reynolds number may be formed as $R_e = VL/\nu$ and molecular friction can be neglected in the limit $R_e \rightarrow \infty$. Similarly the Rossby number is $R_o = V/(fL)$ where $f = 2|\boldsymbol{\Omega}| \sin\phi$ is the Coriolis parameter.

- $$\frac{dc}{dt} \equiv \frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \kappa \frac{\partial^2 c}{\partial x_j \partial x_j}$$

  Expresses conservation of the mass of a passive, non-reactive gas in the atmosphere. The velocity vector $\mathbf{u} = u_i$, and summation applies in any term with a repeated subscript, e.g. $u_j \, \partial c/\partial x_j \equiv \vec{u} \cdot \nabla c$. The molecular diffusivity of the gas in air is κ. If V, L are velocity and length scales for the motion, a Peclet number may be formed as $P = VL/\kappa$

- $D_p + \frac{\partial \omega}{\partial p} = 0$

  Continuity equation in the x, y, p ("isobaric") coordinate system, where $D_p \equiv \nabla_H \cdot \vec{v}_H$ is the divergence of the two velocity components lying in the constant pressure surface ("horizontal divergence")